Tatra Mt. Math. Publ. **40** (2008), 35–42

Tatra Mountains
Mathematical Publications

ON A SUBFAMILY OF DERIVATIVES

Ewa Strońska

ABSTRACT. In this article we introduce and investigate some typical property in the space of all Baire 1 functions. In the class of derivatives this property implies that primitive functions are monotone in some intervals. Moreover, we observe that the quasicontinuity implies this property.

Let $\mathbb{R}$ be the set of reals. As well-known, there are nowhere monotone differentiable functions with bounded derivatives. For example, if $A, B \subset \mathbb{R}$ are countable disjoint dense subsets then there are G_δ sets $U \supset A$ and $V \supset B$ of (Lebesgue) measure zero such that $U \cap B = V \cap A = \emptyset$. In accordance with Zahorski's lemma ([5, Lemma 11] (see also [1, p. 28 Theorem 6.5], and [4]) there are approximately continuous functions $\phi, \psi : \mathbb{R} \to [0,1]$ such that $\phi^{-1}(0) = U$ and $\psi^{-1}(0) = V$. Let

$$h = \phi - \psi \quad \text{and} \quad F(x) = \int_0^x h(t)\,\mathrm{d}t.$$

Observe that

$$h(x) > 0 \qquad \text{for} \quad x \in B$$

and

$$h(x) < 0 \qquad \text{for} \quad x \in A.$$

Since

$$F'(x) = h(x) \qquad \text{for} \quad \text{all} \quad x \in \mathbb{R},$$

F is a differentiable nowhere monotone function with bounded derivative.

Let Δ be the family of all differentiable functions from $\mathbb{R}$ to $\mathbb{R}$ and let Δ_{nm} be the family of all nowhere monotone differentiable functions. Moreover, let $\Delta_{im} = \Delta \setminus \Delta_{nm}$ and let Δ_{am} denote the family of all functions $F \in \Delta$ such that for each open interval I there is an open interval $J \subset I$ on which F is monotone.

2000 Mathematics Subject Classification: 26A15, 26A24, 54C05.
Keywords: derivative, approximate continuity, quasicontinuity, Baire 1 function.

Remark 1. If $F : \mathbb{R} \to \mathbb{R}$ is a differentiable function whose derivative F' is bounded from above or from below, then there are both a real C as well as a monotone differentiable function $G : \mathbb{R} \to \mathbb{R}$ such that

$$F(x) = G(x) + Cx \qquad \text{for all} \quad x \in \mathbb{R}\,.$$

Proof. Assume that F' is strictly bounded from the above by a constant C. For each $x \in \mathbb{R}$ the inequality $F'(x) - C < 0$ is true. Let

$$G(x) = F(x) - Cx \qquad \text{for} \quad x \in \mathbb{R}\,.$$

Since $G' = F' - C < 0$, the function G is strictly monotone. But

$$F(x) = G(x) + Cx \qquad \text{for all} \quad x \in \mathbb{R},$$

so the proof in this case is completed. In the remaining case the proof is similar. □

For a function $f : \mathbb{R} \to \mathbb{R}$ denote by $C(f)$ the set of all its continuity points. All derivatives are of Baire 1 class, so the sets of their continuity points are dense in $\mathbb{R}$. However, they do not have to satisfy condition (a) formulated as follows:

> a function $f : \mathbb{R} \to \mathbb{R}$ satisfies condition (a) if for each open interval I there is an open interval $J \subset I$ for which
>
> $$f(J) \subset [0, \infty) \quad \text{or} \quad f(J) \subset (-\infty, 0].$$

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is called quasicontinuous at a point x if for all positive reals $r, s > 0$ there is an open interval $I \subset (x - r, x + r)$ such that

$$f(I) \subset \big(f(x) - s, f(x) + s\big),$$

(compare [2], [3]).

Remark 2. If a function $f : \mathbb{R} \to \mathbb{R}$ is quasicontinuous then it satisfies condition (a).

Proof. Fix an open interval I and assume that f does not vanish on I. Then there is a point $x \in I$ at which $f(x) \neq 0$. From the quasicontinuity of f at x it follows that there is an open interval $J \subset I$ such that

$$f(J) \subset \left(f(x) - \frac{|f(x)|}{2},\, f(x) + \frac{|f(x)|}{2}\right).$$

Obviously,

$$f(u)f(x) > 0 \qquad \text{for all} \quad u \in J$$

and the proof is completed. □

Observe that the derivative h considered above is bounded and approximately continuous but it does not satisfy condition (a).

Remark 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. If there is an open interval $I \subset cl\big(f^{-1}((0,\infty))\big) \cap cl\big(f^{-1}((-\infty,0))\big)$ then f does not satisfy condition (a).

P r o o f. It suffices to observe that in each open interval $J \subset I$ there are points $u, v \in J$ with $f(u)f(v) < 0$. □

Remark 4. A differentiable function $F \in \Delta_{am}$ if and only if its derivative F' satisfies condition (a).

P r o o f. The proof follows from the observation that a function F is monotone in an open interval if and only if its derivative F' has all values belonging to $[0,\infty)$ or all values belonging to $(-\infty,0]$. □

Remark 5. A differentiable function $F : \mathbb{R} \to \mathbb{R}$ belongs to Δ_{im} if and only if its derivative F' satisfies the following condition

(b) there is an open interval I such that

$$F'(I) \subset [0,\infty) \quad \text{or} \quad F'(I) \subset (-\infty,0].$$

The class of functions satisfying condition (a) is very large. For example, it contains all functions whose all values are nonnegative or all values are nonpositive.

For functions $f, g : \mathbb{R} \to \mathbb{R}$ denote by $\rho(f,g) = \min\big\{1, \sup_{t\in\mathbb{R}} |f(t) - g(t)|\big\}$ their uniform distance. Let $B_{1,a}$ denote the family of all functions $f : \mathbb{R} \to \mathbb{R}$ belonging to Baire 1 class and satisfying condition (a).

Remark 6. Let $f : \mathbb{R} \to \mathbb{R}$ be a function of Baire 1 class. If there is a positive real r with $|f| \geq r$ then $f \in \operatorname{int}(B_{1,a})$, where the interior is considered in the metric space (B_1, ρ).

P r o o f. We start by proving that $f \in B_{1,a}$. Fix an open interval I. Since $f \in B_1$, there is a point $x \in I \cap C(f)$. If $f(x) \geq r$ then there is an open interval $J \subset I$ containing x such that

$$f(J) \subset \left(\frac{r}{2}, \infty\right).$$

If $f(x) \leq -r$ then there is an open interval $J_1 \subset I$ containing x with

$$f(J_1) \subset \left(-\infty, -\frac{r}{2}\right).$$

So f satisfies condition (a). Since for $s \in (0,r)$ each function $g : \mathbb{R} \to \mathbb{R}$ belonging to the ball

$$K(f,s) = \big\{h \in B_1;\, \rho(h,f) < s\big\}$$

satisfies the hypothesis of Remark 6 (so it belongs to $B_{1,a}$), the function $f \in \operatorname{int}(B_{1,a})$. This completes the proof. □

In the last remark the hypothesis that $f \in B_1$ is important. For example, the function

$$f(x) = 1, \qquad \text{at all rationals } x$$

and

$$f(x) = -1, \quad \text{at all irrationals } x$$

is such that $|f| \geq 1$ but f does not satisfy condition (a).

Theorem 1. *The set* $\operatorname{int}(B_{1,a})$ *is dense in the metric space* (B_1, ρ).

Proof. Fix a real $\varepsilon > 0$ and a function $f \in B_1$. For $x \in \mathbb{R}$ let

$$f_1(x) = \max\left\{f(x), \frac{\varepsilon}{3}\right\},$$

$$f_2(x) = \min\left\{f(x), -\frac{\varepsilon}{3}\right\},$$

and

$$g(x) = f_1(x) + f_2(x).$$

Then the function g is of Baire 1 class. If $|f(x)| < \frac{\varepsilon}{3}$ then

$$f_1(x) = \frac{\varepsilon}{3}, \quad f_2(x) = -\frac{\varepsilon}{3}, \quad g(x) = 0$$

and

$$|g(x) - f(x)| < \frac{\varepsilon}{3}.$$

Moreover, if a point u is a continuity point of f and $-\frac{\varepsilon}{3} < f(u) < \frac{\varepsilon}{3}$, then there is an open interval $I_1(u) \ni u$ for which $f\big(I_1(u)\big) \subset (-\frac{\varepsilon}{3}, \frac{\varepsilon}{3})$, and consequently $g\big(I_1(u)\big) = \{0\}$.

If $f(x) \geq \frac{\varepsilon}{3}$, then

$$f_1(x) = f(x), \quad f_2(x) = -\frac{\varepsilon}{3}, \quad g(x) = f(x) - \frac{\varepsilon}{3}$$

and

$$|f(x) - g(x)| = \frac{\varepsilon}{3}.$$

Moreover, if u is a continuity point of f such that $f(u) \geq \frac{\varepsilon}{3}$ then there is an open interval $I_2(u) \ni u$ such that $f\big(I_2(u)\big) \subset \big(\frac{\varepsilon}{4}, \infty\big)$. Observe that for $t \in I_2(u)$ we have

$$f_1(t) \geq \frac{\varepsilon}{3}, \quad f_2(t) = -\frac{\varepsilon}{3}$$

and

$$g(t) \geq \frac{\varepsilon}{3} - \frac{\varepsilon}{3} = 0.$$

If $f(x) \leq -\frac{\varepsilon}{3}$, then

$$f_2(x) = f(x), \quad f_1(x) = \frac{\varepsilon}{3}, \quad g(x) = f(x) + \frac{\varepsilon}{3}$$

and

$$|f(x) - g(x)| = \frac{\varepsilon}{3}.$$

Moreover, if u is a continuity point of f such that $f(u) \leq -\frac{\varepsilon}{3}$, then there is an open interval $I_3(u) \ni u$ such that

$$f\big(I_3(u)\big) \subset \Big(-\infty, -\frac{\varepsilon}{4}\Big).$$

Observe that for $t \in I_3(u)$ we have

$$f_1(t) = \frac{\varepsilon}{3}, \quad f_2(t) \leq -\frac{\varepsilon}{3}$$

and

$$g(t) \leq \frac{\varepsilon}{3} - \frac{\varepsilon}{3} = 0.$$

Let

$$h(t) = g(t) - \frac{\varepsilon}{5} \qquad \text{for} \quad t \in \bigcup I_3(u)$$

and

$$h(t) = g(t) + \frac{\varepsilon}{5} \qquad \text{otherwise on } \mathbb{R}\,.$$

Then h is of Baire 1 class and

$$\rho(h, f) \leq \rho(h, g) + \rho(g, f) \leq \frac{\varepsilon}{5} + \frac{\varepsilon}{3} = \frac{8\varepsilon}{15}.$$

To complete the proof it suffices to prove that each function $k \in B_1$ with $\rho(k, h) < \frac{\varepsilon}{15}$ belongs to $B_{1,a}$. For this fix such a function k and an open interval I. Since the set $C(f)$ is dense, there is a point $u \in C(f) \cap I$. If $f(u) \geq \frac{\varepsilon}{3}$, then for $t \in I_2(u) \cap I$ we have

$$k(t) > h(t) - \frac{\varepsilon}{15} = g(t) + \frac{\varepsilon}{5} - \frac{\varepsilon}{15} > 0.$$

Similarly, we can verify that if $|f(u)| < \frac{\varepsilon}{3}$, then for $t \in I_1(u) \cap I$ we have $k(t) > 0$.

If $f(u) \leq -\frac{\varepsilon}{3}$, then for $t \in I_3(u) \cap I$ we obtain

$$k(t) < h(t) + \frac{\varepsilon}{15} = g(t) - \frac{\varepsilon}{5} + \frac{\varepsilon}{15} < 0.$$

So k satisfies condition (a) and the proof is completed. □

Let $\mathcal{A}$ denote the family of all approximately continuous functions $f : \mathbb{R} \to \mathbb{R}$ and let $\mathcal{A}_a \subset \mathcal{A}$ denote the family of all approximately continuous functions satisfying condition (a).

Theorem 2. *The set $\mathcal{A}_a$ is dense in the metric space $(\mathcal{A}, \rho)$.*

P r o o f. Fix a positive real ε and put

$$f_1(x) = \max\left\{\frac{\varepsilon}{2}, f(x)\right\} \text{ and } f_2(x) = \min\left\{-\frac{\varepsilon}{2}, f(x)\right\} \qquad \text{for} \quad x \in \mathbb{R}\,.$$

Then the functions f_1 and f_2 are approximately continuous, and consequently, the sum $g = f_1 + f_2$ is also approximately continuous. If

$$f(x) \geq \frac{\varepsilon}{2}, \qquad \text{then} \quad f_1(x) = f(x) \quad \text{and} \quad f_2(x) = -\frac{\varepsilon}{2},$$

thus

$$g(x) = f(x) - \frac{\varepsilon}{2}.$$

Similarly, if

$$f(x) \leq -\frac{\varepsilon}{2}, \qquad \text{then} \quad f_2(x) = f(x) \quad \text{and} \quad f_1(x) = \frac{\varepsilon}{2},$$

thus

$$g(x) = f(x) + \frac{\varepsilon}{2}.$$

If

$$-\frac{\varepsilon}{2} < f(x) < \frac{\varepsilon}{2}, \qquad \text{then} \quad f_1(x) = \frac{\varepsilon}{2} \quad \text{and} \quad f_2(x) = \frac{\varepsilon}{2},$$

thus

$$g(x) = 0.$$

Thus $|g(x) - f(x)| \leq \frac{\varepsilon}{2} < \varepsilon$ for each $x \in \mathbb{R}$. We will prove that g satisfies condition (a). For this fix an open interval I. Since f is of Baire 1 class, there is a continuity point $u \in I$ of f. If $|f(u)| < \frac{\varepsilon}{2}$, then there is an open interval $J \subset I$ containing u for which $f(J) \subset \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$. Consequently, for $t \in J$ we have

$$f_1(t) = \frac{\varepsilon}{2}, \;\; f_2(t) = \frac{\varepsilon}{2} \quad \text{and} \quad g(t) = f_1(t) + f_2(t) = 0,$$

thus

$$g(J) \subset (-\infty, 0].$$

If $f(u) \geq \frac{\varepsilon}{2}$, then there is an open interval $J \subset I$ containing u with $f(J) \subset \left(\frac{\varepsilon}{4}, \infty\right)$. Then for $t \in J$ we have

$$f_1(t) \geq \frac{\varepsilon}{2}, \;\; f_2(t) = -\frac{\varepsilon}{2} \quad \text{and} \quad g(t) \geq 0,$$

thus

$$g(J) \subset [0, \infty).$$

Similarly, if $f(u) \leq -\frac{\varepsilon}{2}$, then there is an open interval $J \subset I$ containing u with $g(J) \subset (-\infty, 0]$. So the function g satisfies condition (a) and the proof is completed. □

Since the families $\mathcal{A}$ of all approximately continuous functions and Q of all quasicontinuous functions are closed with respect to the uniform convergence, the subspace $(Q \cap \mathcal{A}, \rho)$ is closed in the space $(\mathcal{A}, \rho)$.

Remark 7. The set $Q \cap \mathcal{A}$ is nowhere dense in the space $(\mathcal{A}, \rho)$.

P r o o f. Fix a real $\varepsilon > 0$ and a function $f \in Q \cap \mathcal{A}$. There is a continuity point x of f. Let $I \ni x$ be an open interval such that $f(I) \subset \big(f(x) - \frac{\varepsilon}{3}, f(x) + \frac{\varepsilon}{3}\big)$ and let $E \subset I$ be a nowhere dense F_σ-set belonging to the density topology and containing x. From Zahorski's lemma ([5, Lemma 11]) it follows that there is an approximately continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $g(\mathbb{R}) = [0,1]$, $f(x) = 1$ and $g(\mathbb{R} \setminus E) = \{0\}$. Let $h = f + \frac{2\varepsilon}{3} \cdot g$. Then h is approximately continuous,

$$h(x) = f(x) + \frac{2\varepsilon}{3}, \quad \text{and} \quad h(t) = f(t) < f(x) + \frac{\varepsilon}{3} \qquad \text{for} \quad t \in \mathbb{R} \setminus E.$$

So h is not quasicontinuous at x. Evidently, $|f - h| = \frac{2\varepsilon}{3} \cdot |g| \le \frac{2\varepsilon}{3} < \varepsilon$, and the proof is completed. □

Theorem 3. *Let $f : \mathbb{R} \to \mathbb{R}$ be an approximately continuous function such that $f\big(C(f)\big) \subset [0,\infty)$ or $f\big(C(f)\big) \subset (-\infty, 0]$. Then f belongs to the uniform closure of the interior* $\operatorname{int}(\mathcal{A}_a)$ *considered in the space $(\mathcal{A}, \rho)$.*

P r o o f. The proof is similar to the proof of Theorem 1. If we assume that

$$f\big(C(f)\big) \subset [0,\infty),$$

then there do not exist the intervals $I_3(u)$ and for all $t \in \mathbb{R}$ we put

$$h(t) = g(t) + \frac{\varepsilon}{5}.$$

If $f\big(C(f)\big) \subset (-\infty, 0]$ the reasoning is analogous. □

Problems.

(1) Is the interior $\operatorname{int}(\mathcal{A}_a)$ of the set $\mathcal{A}_a$ dense in the metric space $(\mathcal{A}, \rho)$?
Consider the metric space (Δ', ρ) of all derivatives with the metric ρ and the set Δ'_a of all derivatives satisfying condition (a).

(2) Is the interior $\operatorname{int}(\Delta'_a)$ dense in (Δ', ρ)?

(3) Is the set Δ'_a dense in (Δ', ρ)?

Acknowledgement. The author is grateful to the referee for his/her corrections of Theorem 2.

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Received November 8, 2006

Institute of Mathematics
Kazimierz Wielki University
Plac Weyssenhoffa 11
PL–85-072 Bydgoszcz
POLAND
E-mail: stronska@neostrada.pl