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ON VARIOUS DEFINITIONS OF DENSITY (SURVEY)

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ABSTRACT. This is an incomplete collection of elementary considerations, remarks and questions on the notion of density.

1. Usual densities. Isobaricity

We deal with densities of subsets of $\mathbb{N} = \{1, 2, 3, \dots\}$.

The most frequently encountered densities are the asymptotic (or natural) density, the Schnirelmann density and the logarithmic density [see Definition 1 below]. To each density concept, an upper density and a lower density are associated, both defined for *every* subset of $\mathbb{N}$. In the case where these two densities (upper and lower) coincide, we say that the subset of $\mathbb{N}$ "has density". This happens, in general, only for *some* subsets of $\mathbb{N}$.

Definition 1. Let $A \subseteq \mathbb{N}$. For real $x \geqslant 1$, let $A(x) = \big|A \cap [1, x]\big|$ where $|S|$ denotes the cardinality of the finite set S. The function $A(x)$, $x \geqslant 1$, is called *the counting function* of the set A. We define

– the *Schnirelmann density*

$$\sigma A = \inf_{n \in \mathbb{N}} \frac{A(n)}{n},$$

– the *lower asymptotic density*

$$\underline{d} A = \liminf_{x \to \infty} \frac{A(x)}{x},$$

– the *upper asymptotic density*

$$\overline{d} A = \limsup_{x \to \infty} \frac{A(x)}{x},$$

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– the *lower logarithmic density*

$$\underline{\delta} A = \liminf_{x\to\infty}\Bigg(\sum_{a\in A,\, a\leqslant x}\frac{1}{a}\Bigg)\Bigg(\sum_{a\leqslant x}\frac{1}{a}\Bigg)^{-1} = \liminf_{x\to\infty}\Bigg(\sum_{a\in A,\, a\leqslant x}\frac{1}{a}\Bigg)(\ln x)^{-1}\,,$$

– the *upper logarithmic density*

$$\overline{\delta} A = \limsup_{x\to\infty}\Bigg(\sum_{a\in A,\, a\leqslant x}\frac{1}{a}\Bigg)\Bigg(\sum_{a\leqslant x}\frac{1}{a}\Bigg)^{-1} = \limsup_{x\to\infty}\Bigg(\sum_{a\in A,\, a\leqslant x}\frac{1}{a}\Bigg)(\ln x)^{-1}\,.$$

We have [HR, p. 241]

$$0 \leqslant \sigma A \leqslant \underline{d} A \leqslant \underline{\delta} A \leqslant \overline{\delta} A \leqslant \overline{d} A \leqslant 1 \tag{1}$$

for every set $A \subseteq \mathbb{N}$. If $\underline{d}A = \overline{d}A$, then we say that the set A has asymptotic density $dA = \underline{d}A$. The same holds for the logarithmic density. It is proved in [M], using methods of [MT], that there exist sets with arbitrary prescribed values of densities satisfying the relation (1).

The Schnirelmann density defined above is a “lower” density. One could define an upper Schnirelmann density by

$$\overline{\sigma} A = \sup_{n\in\mathbb{N}} \frac{A(n)}{n}$$

and relation (1) would be completed into:

$$0 \leqslant \sigma A \leqslant \underline{d} A \leqslant \overline{d} A \leqslant \overline{\sigma} A \leqslant 1\,.$$

The reason for which $\overline{\sigma} A$ is not popular is that very often it is equal to 1: $\overline{\sigma} A = 1$ whenever σA is strictly positive. Precisely, for any set A one has either $\sigma A = 0$ or $\overline{\sigma} A = 1$.

The density concept is a very simple and well understood concept. It should be noted that usually, for instance in Number Theory, it appears like a conclusion summarizing calculations and giving account of a more precise inequality or estimation. For instance, the prime number theorem, or simply the Tchebychev inequality, implies that the asymptotic density of the set of primes is zero.

After a rapid look at definitions, one may conclude that there are two types of densities:

– *isobaric* densities, where the weight of each element of the set is the same;

– *non isobaric* (or *weighted*) densities where the contributions of elements are different.

Thus one would qualify the Schnirelmann density and the asymptotic densities as isobaric, the logarithmic densities as non isobaric. In further examples of densities appearing below, the uniform and the exponential densities [Definitions 2 and 3] should be isobaric and the analytic densities [Definition 4] should be qualified as non isobaric.

Let us end this section with three comments.

(I) In social choice problems it seems more *democratic* to consider only isobaric densities.

(II) In the preceding considerations, isobaricity is deduced from the appearance of the *counting function* in the formula defining the density. Thus each element of the set is counted with weight one.

But perhaps this is not an adequate criterion. Look at the Schirelmann density. If 1 is a member of the set A, then it contributes to $\frac{A(n)}{n}$ with the same weight as any other member of A. But if 1 doesn't belong to A, this implies that $\sigma A = 0$. Moreover $1 \in A$ implies $\frac{A(n)}{n} > 0$ for all n and, in this case, for having $\sigma A = 0$ it is necessary that all integers in a set of upper asymptotic density 1 do not belong to A: $1 \in A \Rightarrow (\sigma A = 0 \Leftrightarrow \underline{d}A = 0)$.

(III) *Size and position.* It is clear that in the weighted case, density depends on the position of the elements of the set A in the ordered sequence of natural numbers

$$1, 2, 3, 4, \ldots$$

But we would like to emphasize that this is also true in the isobaric case and constitutes a main difference between finiteness and infiniteness. *Finite sets*: Let N be a fixed positive integer. To any subset A of $\{1, 2, \ldots, N\}$ we may attribute as measure of his size (density) the number

$$m(A) = \frac{|A|}{N} \tag{2}$$

and this number is independent of the positions of the elements of A among $1, 2, \ldots, N$. *Infinite sets*: Let $A \subseteq \mathbb{N}$. Consider the asymptotic density which extends the measure in (2). Except for the cases where A is finite or his complement $A^c = \mathbb{N} \setminus A$ is finite (and then $dA = 0$ or 1, respectively, is a value *independent* of the position of the elements of A in the sequence $1, 2, 3, \ldots$), in all other cases the densities $\underline{d}A$ and $\overline{d}A$ depend *entirely* on the position of the elements of A among $1, 2, \ldots$. The sets A and A^c, if they are not finite, have the same cardinality ($\aleph_0$) as $\mathbb{N}$. And moreover, given two real numbers α, β, $0 \leqslant \alpha \leqslant \beta \leqslant 1$, it is possible to find a permutation $\tau \colon \mathbb{N} \to \mathbb{N}$ such that $\underline{d}\tau(A) = \alpha$ and $\overline{d}\tau(A) = \beta$, where these densities are defined as the lower and the upper limits of

$$\frac{1}{n}\Big|\{a \in A \colon \tau(a) \leqslant n\}\Big|.$$

For results and bibliography on density preserving permutations, see [O].

2. Uniform density

Let $A \subseteq \mathbb{N}$. For $1 \leqslant x \leqslant y$, let $A(x,y) = \left|A \cap [x,y]\right|$. Thus the counting function satisfies $A(x) = A(1,x)$. Let s be a real number, $s \geqslant 1$. We define

$$\alpha_s = \liminf_{k\to\infty} A(k+1, k+s)$$

and

$$\alpha^s = \limsup_{k\to\infty} A(k+1, k+s)\,.$$

Both α_s and α^s are integers belonging to $\{0,1,\cdots,\lfloor s\rfloor\}$, $\alpha_s \leqslant \alpha^s$. Each one, divided by s, *tends to a limit*, when s tends to infinity. A proof of this fact can be found in [Ri] or in [ST].

Definition 2. The *lower uniform density* of $A \subseteq \mathbb{N}$ is defined as

$$\underline{u}A = \lim_{s\to\infty} \frac{\alpha_s}{s}$$

and its *upper uniform density* as

$$\overline{u}A = \lim_{s\to\infty} \frac{\alpha^s}{s}\,.$$

For any set $A \subseteq \mathbb{N}$, we have

$$0 \leqslant \underline{u}A \leqslant \underline{d}A \leqslant \overline{d}A \leqslant \overline{u}A \leqslant 1\,. \tag{3}$$

The Schnirelmann density σA, depending upon the set A, may be smaller, equal or bigger than $\underline{u}A$. The same is true for $\overline{\sigma}A$ and $\overline{u}A$.

Uniform density is used in many problems. For instance, it allowed to strengthen results on diophantine equations (see, [BF], [Ri]). See also, [GLS] for a study of uniform density.

3. Axioms

Is there some convenient axiomatic definition of density? The *obligatory* axioms should be the following ones.

Let $\mathcal{P}(\mathbb{N}) = \{A\colon A \subseteq \mathbb{N}\}$ be the power set of $\mathbb{N}$. Let $\mathcal{D} \subseteq \mathcal{P}(\mathbb{N})$, $\emptyset \in \mathcal{D}$ and $\mathbb{N} \in \mathcal{D}$. A *density function* is a function

(A.1) $f\colon \mathcal{D} \to [0,1]$
such that

(A.2) $f(\emptyset) = 0$, $f(\mathbb{N}) = 1$
and

(A.3) $A \in \mathcal{D}$, $B \in \mathcal{D}$, $A \subseteq B \Rightarrow\ f(A) \leqslant f(B)$.

One could (or should?) restrict the axioms to the preceding ones; or add further axioms suggested by some properties of densities. For instance:

(A.4) Invariability by translation: For all $A \in \mathcal{D}$, $A + t = \{a + t \colon a \in A\} \in \mathcal{D}$ and $f(A + t) = f(A)$.

(A.5) Homothetic transformation: For all $A \in \mathcal{D}$ and $t \in \mathbb{N}$, $t \cdot A = \{ta \colon a \in A\} \in \mathcal{D}$ and $f(t \cdot A) = \frac{1}{t} f(A)$.

(A.6) Finiteness: For any finite subset F of $\mathbb{N}$, we have $F \in \mathcal{D}$ and $f(F) = 0$.

Some of these properties, and especially the last one, should be theorems if isobaricity is assumed. But *how* can one *introduce isobaricity axiomatically*?

All upper and lower densities previously defined satisfy all these properties, except Schnirelmann density which does not satisfy (A.4) and (A.5).

Much work has been done in relation with the following axiom of *finite additivity*:

(A.7) If $A \in \mathcal{D}$, $B \in \mathcal{D}$ are such that $A \cap B = \emptyset$, then $A \cup B \in \mathcal{D}$ and $f(A \cup B) = f(A) + f(B)$.

Notice that a density satisfying (A.6) cannot be σ-additive because, for all $n \in \mathbb{N}$, $f(n) = 0$, and this would imply

$$f\left(\bigcup_{n=1}^{\infty} \{n\}\right) = \sum_{n=1}^{\infty} 0 = 0$$

which contradicts the second axiom (A.2). Asymptotic, logarithmic and uniform densities, *when they exist*, satisfy (A.7). The sentence, "when they exist" means that $\mathcal{D}$ is a proper subset of $\mathcal{P}(\mathbb{N})$, restricted to those sets A for which upper and lower densities are equal.

Motivated by classical measure theory, one may consider that "perfect" densities satisfy (A.7) *and* $\mathcal{D} = \mathcal{P}(\mathbb{N})$.

It was proved by Banach [Ba] in a general context that such a density exists. Searching an effective construction, Buck [Bu] proved that there exists a density f satisfying (A.7) with $\mathcal{D} = \mathcal{P}(\mathbb{N})$ and such $f(A) = dA$ for all sets $A \subseteq \mathbb{N}$ for which $\underline{d}A = \overline{d}A$. For a variant of this result, using non standard analysis in proof, see [Ro, p. 64–67].

The problem with the three mentioned "measure" densities is that they are quite ineffective ([Ba], [Ro]) or not easy to handle ([Bu]). So classical definitions remain popular in spite of the fact that $\mathcal{D} \subsetneqq \mathcal{P}(\mathbb{N})$.

Freedman and Sember [FS] introduced an axiomatization adapted to the lower asymptotic density with $\mathcal{D} = \mathcal{P}(\mathbb{N})$ of course. Here are the axioms.

(FS-1) $f \colon \mathcal{P}(\mathbb{N}) \to [0, 1]$ is such that

(FS-2) $|A \triangle B| < \infty \Rightarrow f(A) = f(B)$, (FS-3) $A \cap B = \emptyset \Rightarrow f(A) + f(B) \leqslant f(A \cup B)$,

(FS-4) $f(A) + f(B) \leqslant 1 + f(A \cap B)$,

(FS-5) $f(\mathbb{N}) = 1$.

Then they *define* the upper density $\overline{f}$ by

$$\overline{f}(A) = 1 - f(\mathbb{N} \backslash A) \tag{4}$$

and they prove that

$$\begin{aligned}
&A \subseteq B \Rightarrow f(A) \leqslant f(B) \text{ and } \overline{f}(A) \leqslant \overline{f}(B)\,,\\
&f(\emptyset) = \overline{f}(\emptyset) = 0\,,\ \overline{f}(\mathbb{N}) = 1\,,\\
&|A \triangle B| < \infty \Rightarrow \overline{f}(A) = \overline{f}(B)\,,\\
&f(A) \leqslant \overline{f}(A)\,,\\
&\overline{f}(A) + \overline{f}(B) \geqslant \overline{f}(A \cup B)\,.
\end{aligned}$$

The following example of density shows that relation (4) between lower and upper densities does not hold in some cases.

4. The exponential density

We obtain a family of isobaric densities by the following definition.

Definition 3. Let h be a positive unbounded increasing function defined on $\mathbb{R}_+ = [0, +\infty[$. We define the *lower h-density* of the set $A \subseteq \mathbb{N}$ by

$$\underline{\varepsilon}_h(A) = \liminf_{n \to \infty} \frac{h\big(A(n)\big)}{h(n)}$$

and its *upper h-density* by

$$\overline{\varepsilon}_h(A) = \limsup_{n \to \infty} \frac{h\big(A(n)\big)}{h(n)}\,.$$

The function $h(x) = \log x$ is frequently used and the corresponding h-density is called *exponential density*, here denoted by ε.

The function $h(x) = x^\alpha$ is not so interesting as it gives $\underline{\varepsilon}_h(A) = (\underline{d}A)^\alpha$ and similarly for upper densities. The exponential density ε acts as a magnifying glass on sets with asymptotic density zero. If $A = \{1^k, 2^k, 3^k, \dots\}$ then $\varepsilon A = \frac{1}{k}$. Moreover, the preceding example is the reason for the name of the density ("exponential") as the exponent k determines εA.

Let us use the notation $\mathbb{N}^{(k)} = \{1^k, 2^k, 3^k, \dots\}$. Taking

$$A = \bigcup_{n=1}^{\infty} \Big\{ \big([a_n, b_n[\cap \mathbb{N}^{(2)} \big) \cup \big([b_n, a_{n+1}[\cap \mathbb{N}^{(3)} \big) \Big\} \tag{5}$$

with $a_n < b_n < a_{n+1}$ (for all n), a_n, b_n integers, and b_n (resp. a_{n+1}) sufficiently large with respect to a_n (resp. b_n), we can obtain a set A with

$$\underline{\varepsilon} A = \frac{1}{3}\,, \quad \overline{\varepsilon} A = \frac{1}{2}\,.$$

We observe that:

$$\underline{d} A > 0 \Rightarrow \underline{\varepsilon} A = 1\,; \quad \overline{d} A > 0 \Rightarrow \overline{\varepsilon} A = 1\,.$$

Note also that the lower exponential density is not sub-additive (as is the lower asymptotic density): take $A = \mathbb{N}^{(2)}$, $B = A + 1$; we have

$$\varepsilon A = \varepsilon B = \varepsilon (A \cup B) = \frac{1}{2}\,.$$

Another interesting fact is that using a formula similar to (5), we can construct a set A such that

$$\underline{d} A < \underline{\varepsilon} A < \overline{d} A < \overline{\varepsilon} A\,.$$

By choosing b_n and a_{n+1} sufficiently large in each step, the set

$$A = \bigcup_{n=1}^{\infty} \Big\{ \big([a_n, b_n[\cap (2.\mathbb{N}) \big) \cup \big([b_n, a_{n+1}[\cap \mathbb{N}^{(3)} \big) \Big\} \tag{6}$$

has densities

$$\underline{d} A = 0\,, \quad \underline{\varepsilon} A = \frac{1}{3}\,, \quad \overline{d} A = \frac{1}{2}\,, \quad \overline{\varepsilon} A = 1\,.$$

The above mentioned facts put the question what would be an appropriate axiomatic definition for upper and lower densities and for relations between them.

Also the densities associated to the function $h(x) = \ln(\ln x)$ or to other *slowly varying* functions h (that is, for every $t > 0$, $\frac{h(tx)}{h(x)} \to 1$ as $x \to \infty$) are interesting.

5. Weighted densities

Relatively to a nonzero "weight" function $f\colon \mathbb{N} \to [0, +\infty[$, the *$f$-weighted densities* are defined as the lower and the upper limits of

$$\Bigg(\sum_{a \in A,\ a \leqslant n} f(a) \Bigg) \Bigg(\sum_{k \leqslant n} f(k) \Bigg)^{-1}$$

as n tends to infinity. Usually one takes a function satisfying

$$\sum_{n=1}^{\infty} f(n) = +\infty\,.$$

See also [RV] for other conditions.

The special case $f(n) = n^\alpha$, $\alpha \geqslant -1$, generalizes both the asymptotic density $(\alpha = 0)$ and the logarithmic density $(\alpha = -1)$. Let us denote by $\underline{d}_\alpha A$ and $\overline{d}_\alpha A$ the lower and the upper densities of the set A, where the weight function is $f(n) = n^\alpha$, $\alpha \geqslant -1$. Let α, β be two real numbers, $-1 \leqslant \alpha < \beta$. The relation

$$0 \leqslant \underline{d}_\beta A \leqslant \underline{d}_\alpha A \leqslant \overline{d}_\alpha A \leqslant \overline{d}_\beta A \leqslant 1 \tag{7}$$

is true for all $A \subseteq \mathbb{N}$ and generalizes the relation (1). It can be obtained as a corollary of a result by C. T. Rajagopal [Ra; Theorem 3, p. 161] on limits. See also [Po]. Another important fact is that the existence of $d_\alpha A$ for some $\alpha > -1$, implies the existence of $d_\beta A$ for all $\beta \geqslant -1$. (The two values are equal in virtue of (7).) This is proved in [FG-A].

All weighted densities satisfy

$$\underline{d}_\alpha A^c = 1 - \overline{d}_\alpha A\,, \quad \overline{d}_\alpha A^c = 1 - \underline{d}_\alpha A\,,$$

where $A^c = \mathbb{N} \setminus A$. Also if $C = A \cup B$ with $A \cap B = \emptyset$, we have

$$\underline{d}_\alpha A + \underline{d}_\alpha B \leqslant \underline{d}_\alpha C \leqslant m \leqslant M \leqslant \overline{d}_\alpha A + \overline{d}_\alpha B\,,$$

where $m = \min E$, $M = \max E$, $E = \{\underline{d}_\alpha A + \overline{d}_\alpha B,\ \overline{d}_\alpha A + \underline{d}_\alpha B\}$.

6. Other densities. Conclusion

6.1. Here is the definition of the analytic (or Dirichlet) density.

Definition 4. The *analytic densities* of a set $A \subseteq \mathbb{N}$ are defined as the lower and the upper limits, as $\sigma \to 1^+$, of

$$\zeta(\sigma)^{-1} \sum_{\alpha \in A} a^{-\sigma}, \tag{8}$$

where $\zeta(\sigma) = \sum_{k \in \mathbb{N}} k^{-\sigma}$, $\sigma > 1$, is the Riemann dzeta function. Since

$$\lim_{\sigma \to 1^+} (\sigma - 1)\zeta(\sigma) = 1\,,$$

instead of considering the upper and lower limits of expression (8) one can equivalently consider the upper and lower limits of

$$(\sigma - 1) \sum_{a \in A} a^{-\sigma}.$$

It is known [T] that the analytic density exists if and only if the logarithmic density δA exists. In that case they are equal.

6.2. *Conditional* (or *relative*) density with respect to a subset $\mathbb{H}$ of $\mathbb{N}$ can be defined. See [FL] for an application. In the case of asymptotic density, the conditional ones are the lower and the upper limits, as n goes to infinity, of

$$\left|(A \cap \mathbb{H}) \cap [1, n]\right| \mathbb{H}(n)^{-1} .$$

6.3. *Restricted density* can be defined according to a given subset $\mathbb{M} = \{m_1 < m_2 < \dots\}$ of $\mathbb{N}$. In the asymptotic case it is defined as the upper or lower limit of $m_n^{-1} A(m_n)$, as n tends to infinity.

6.4. Density can be associated to other discrete commutative (countable) semi-groups. Thus we can define the density of a subset of the cartesian product $\mathbb{N}^k$, of a collection of finite subsets of $\mathbb{N}$, or of a subset of an arithmetic semi-group. See [DE], [Po], [Rz].

6.5. It is worthwhile to notice that some densities, but not all, appear as limits in the image of the characteristic function of A by a "regular" matrix transformation. For instance, the asymptotic densities of a set A are the lower and upper limits of y_n $(n \to \infty)$ where

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \end{pmatrix}$$

is the column vector given by the relation

$$MX = Y,$$

where

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix}$$

is the column vector associated to A with

$$x_n = 1 \text{ if } n \in A, \quad x_n = 0 \text{ if } n \notin A$$

and M is the infinite "regular" matrix

$$M = \begin{pmatrix} 1 & & \\ 1/2 & 1/2 & \\ 1/3 & 1/3 & 1/3 \\ \dots & & \end{pmatrix} .$$

For the definition of the regularity of (infinite) matrices and links with summability methods we refer to [Pe]. Thus densities are of two kinds: those associated to regular matrices and the others ...

Conclusion. Further aspects related to measure theory, to mean values, to summability methods and to applications to social science were presented during this workshop. Other features and open problems should lead to future investigations. As no axiomatics seems to be imposed, this may give rise to unexpected developments inside this elementary subject.

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