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SUMS OF $\mathscr{T}_{\mathcal{I}}$-QUASI-CONTINUOUS FUNCTIONS

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ABSTRACT. For each finite family of cliquish functions $\mathfrak{A}$ we can find a Lebesgue function α such that $f + \alpha$ is strong Świątkowski and $\mathscr{T}_{\mathcal{I}}$-quasi-continuous for every $f \in \mathfrak{A}$. Consequently, each cliquish function is the sum of two $\mathscr{T}_{\mathcal{I}}$-quasi-continuous strong Świątkowski functions. This result is a strengthened version of theorems proved earlier by H. W. Pu & H. H. Pu, Z. Grande, and A. Maliszewski, and it solves a problem posed by K. Flak.

1. Preliminaries

1.1 Notation

The real line $(-\infty, \infty)$ is denoted by $\mathbb{R}$ and the set of positive integers by $\mathbb{N}$. The word *function* means a mapping from $\mathbb{R}$ into $\mathbb{R}$. The words *measure*, *summable*, a symbol like $\int_A f$, etc. always refer to the Lebesgue measure and integral in $\mathbb{R}$. The Euclidean metric in $\mathbb{R}$ will be denoted by ϱ. If $A \subset \mathbb{R}$ is nonempty, then let $\operatorname{diam} A$ be its diameter; i.e., $\operatorname{diam} A = \sup\{|x - y| : x, y \in A\}$.

Let f be a function and let $A \subset \mathbb{R}$ be nonempty. The *oscillation of f on A* will be denoted by $\omega(f, A)$; i.e., $\omega(f, A) = \sup f[A] - \inf f[A]$. Similarly, the *oscillation of f at a point $x \in \mathbb{R}$* will be denoted by $\omega(f, x)$; i.e., $\omega(f, x) = \lim_{\delta \to 0^+} \omega\bigl(f, [x - \delta, x + \delta]\bigr)$. The set of points of continuity of f will be denoted by $\mathrm{C}(f)$.

Throughout the paper, $\mathscr{B}$ denotes the family of all subsets of $\mathbb{R}$ possessing the Baire property, and $\mathcal{I}$ denotes the σ-ideal of all meager sets. For brevity, we introduce the notation $A \sqsubset B$ for $A \setminus B \in \mathcal{I}$.

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1.2 Topology $\mathscr{T}_{\mathcal{I}}$

The notion of the $\mathcal{I}$-density point was introduced in [11]. Its further properties were studied in [14].

We say that 0 is an *$\mathcal{I}$-density point* of a set $A \in \mathscr{B}$ iff for every increasing sequence (n_m) of positive integers there exists a subsequence (n_{m_p}) such that

$$\chi_{(n_{m_p} \cdot A) \cap [-1,1]} \longrightarrow \chi_{[-1,1]} \quad \text{as } p \to \infty$$

except for a meager set. (The symbol $n_{m_p} \cdot A$ stands for $\{n_{m_p} y \colon y \in A\}$.) A point x_0 is an $\mathcal{I}$-density point of a set $A \in \mathscr{B}$ iff 0 is an $\mathcal{I}$-density point of $-x_0 + A$. (Similarly, $-x_0 + A = \{-x_0 + y \colon y \in A\}$.)

It is well-known that the family of all sets $A \in \mathscr{B}$ with the property that each $x \in A$ is an $\mathcal{I}$-density point of A is a topology in $\mathbb{R}$ finer than the Euclidean one ([11]). We will denote this topology by $\mathscr{T}_{\mathcal{I}}$.

1.3 Quasi-continuity and cliquishness

The notion of quasi-continuity was introduced in 1932 by S. Kempisty [4], and the notion of cliquishness—by H. Thielman [13].

Let $\mathscr{T}$ be a topology in $\mathbb{R}$ and $f \colon \mathbb{R} \to \mathbb{R}$. We say that f is *$\mathscr{T}$-quasi-continuous* (*$\mathscr{T}$-cliquish*) at a point $x \in \mathbb{R}$, if for each $\varepsilon > 0$ and each $\mathscr{T}$-open set $U \ni x$ we can find a nonempty $\mathscr{T}$-open set $V \subset U$ such that $\omega\big(f, \{x\} \cup V\big) < \varepsilon$ (such that $\omega(f, V) < \varepsilon$, respectively). We say that f is *$\mathscr{T}$-quasi-continuous* (*$\mathscr{T}$-cliquish*), if it is $\mathscr{T}$-quasi-continuous ($\mathscr{T}$-cliquish) at each point $x \in \mathbb{R}$. If $\mathscr{T}$ is the Euclidean topology in $\mathbb{R}$, then we will write simply quasi-continuous and cliquish. It is well-known that a function f is cliquish iff it is *pointwise discontinuous*; i.e., iff the set $\mathrm{C}(f)$ is dense in $\mathbb{R}$.

1.4 Lebesgue functions

We say that $x \in \mathbb{R}$ is a *Lebesgue point* of a function $\alpha \colon \mathbb{R} \to \mathbb{R}$, if α is locally summable at x and $\lim\limits_{\delta \to 0} \int_x^{x+\delta} \big|\alpha - \alpha(x)\big| / \delta = 0$. We say that α is a *Lebesgue function*, if each $x \in \mathbb{R}$ is a Lebesgue point of α. Recall that each Lebesgue function is an approximately continuous derivative.

1.5 Strong Świątkowski functions

We say that a function $\varphi \colon \mathbb{R} \to \mathbb{R}$ is *strong Świątkowski* ([7]), if whenever $\alpha < \beta$ and $\big(f(\alpha) - y\big)\big(f(\beta) - y\big) < 0$, there is an $x_0 \in (\alpha, \beta) \cap \mathrm{C}(f)$ with $f(x_0) = y$. Recall that strong Świątkowski functions are both Darboux and quasi-continuous. (See, e.g., [7] or [9].)

2. Introduction

In 1987 H. W. Pu and H. H. Pu proved the following theorem [12].

THEOREM 2.1. *Let $f_1, \dots, f_k$ be Baire one functions. There exists a Baire one function α such that $f_i + \alpha$ is Darboux for each $i \in \{1, \dots, k\}$.*

This theorem was generalized in 1992 by Z. Grande [3].

THEOREM 2.2. *Let $f_1, \dots, f_k$ be cliquish functions. There exists a Baire one function α such that $\alpha \neq 0$ only on a null set and $f_i + \alpha$ is Darboux for each $i \in \{1, \dots, k\}$.*

In 1996 A. Maliszewski showed that we can require even more [10, Theorem 4]. (See also [9, Theorem I.3.2].) Clearly, we cannot require that $\alpha \neq 0$ only on a null set.

THEOREM 2.3. *Let $f_1, \dots, f_k$ be cliquish functions and $\tau > 0$. There exists a Lebesgue function α such that $f_i + \alpha$ is strong Świątkowski*[1] *for each $i \in \{1, \dots, k\}$, $|\alpha| < \sup\{\omega(f_i, x) \colon i \in \{1, \dots, k\},\, x \in \mathbb{R}\} + \tau$ on $\mathbb{R}$, and $\mathrm{C}(\alpha) \supset \bigcap_{i=1}^{k} \mathrm{C}(f_i)$.*

K. Flak proved in her PhD thesis [1] that each $\mathscr{T}_{\mathcal{I}}$-quasi-continuous function is quasi-continuous and that the converse is not true, while the notions of $\mathscr{T}_{\mathcal{I}}$-cliquishness and cliquishness coincide. So, it is natural to ask whether we can strengthen the conclusion of Theorem 2.3, adding the requirement that each sum $f_i + \alpha$ be moreover $\mathscr{T}_{\mathcal{I}}$-quasi-continuous. We will show we can. As a corollary we obtain that each cliquish function is the sum of two $\mathscr{T}_{\mathcal{I}}$-quasi-continuous functions, which answers a question posed by K. Flak [2] in the positive.

3. Auxiliary results

We will need the following lemmas. The first of them is a part of [6, Lemma 1]. (See also [5, Theorem 1].)

LEMMA 3.1. *Suppose that $U \in \mathscr{B}$ and x is an $\mathcal{I}$-density point of U. Then there are $\eta > 0$ and $r \in \mathbb{N}$ such that for each $\delta \in (0, \eta)$, we can find $p, q \in \{1, \dots, r\}$ with*

$$\left(x - \frac{p}{r}\,\delta,\; x - \frac{p-1}{r}\,\delta\right) \cup \left(x + \frac{q-1}{r}\,\delta,\; x + \frac{q}{r}\,\delta\right) \sqsubset U\,.$$

[1] Actually A. Maliszewski did not use this notion, which was introduced later in [7].

The proof of the next lemma is straightforward. (See also [9, Lemma I.3.1], [10, Lemma 3], and [8, Lemma 3.3].)

Lemma 3.2. *Let $k \geq 1$, I be a compact interval, functions $g_1, \dots, g_k$ be cliquish, $K \subset \mathbb{R}$ be nowhere dense and closed, $\tau > \max\{\omega(g_i, I) \colon i \in \{1, \dots, k\}\}$, and $L, \varepsilon > 0$. There exists a nowhere dense perfect set $F \subset I \cap \bigcap_{i=1}^{k} \mathrm{C}(g_i) \setminus K$ and a continuous function α such that $|\alpha| < L + \tau$ on $\mathbb{R}$, $\alpha = 0$ on $K \cup (\mathbb{R} \setminus I)$, $\int_I |\alpha| < \varepsilon$, and $(g_i + \alpha)[F] \supset \big[\inf g_i[I] - L, \sup g_i[I] + L\big]$ for each i.*

4. Main results

The proof of our main result is patterned on that of [10, Theorem 4].

Theorem 4.1. *Let $k \geq 1$, $f_1, \dots, f_k$ be cliquish functions and $\tau > 0$. There exists a Lebesgue function α such that $f_i + \alpha$ is strong Świątkowski and $\mathscr{T}_{\mathcal{I}}$-quasi-continuous for each i, $|\alpha| < \sup\{\omega(f_i, x) \colon i \in \{1, \dots, k\},\, x \in \mathbb{R}\} + \tau$ on $\mathbb{R}$, and $\mathrm{C}(\alpha) \supset \bigcap_{i=1}^{k} \mathrm{C}(f_i)$.*

Proof. First set $\tau_0 = \sup\{\omega(f_i, x) \colon i \in \{1, \dots, k\},\, x \in \mathbb{R}\} + \tau/2$, $A_0 = \emptyset$, and $\tau_n = \tau/5^n$, $A_n = \bigcup_{i=1}^{k} \{x \in \mathbb{R} \colon \omega(f_i, x) \geq \tau_n\}$ for $n \geq 1$. Then each set A_n is closed, and since $f_1, \dots, f_k$ are cliquish, it is nowhere dense. Let $C = \bigcap_{i=1}^{k} \mathrm{C}(f_i)$. We will construct by induction a sequence of nowhere dense closed sets, (F_n), and a sequence of Lebesgue functions, (α_n). Put $\alpha_0 = 0$ and $F_0 = \emptyset$. Fix an $n \in \mathbb{N}$.

Find a family, $\{I_{n,m} \colon m \in \mathbb{N}\}$, consisting of nonoverlapping compact intervals, such that $\bigcup_{m \in \mathbb{N}} I_{n,m} = \mathbb{R} \setminus A_n$ and each $x \notin A_n$ belongs to the interior of the union $I_{n,m} \cup I_{n,p}$ for some $m, p \in \mathbb{N}$. For each m, since $I_{n,m}$ is a compact set, $\omega(f_i, x) < \tau_n$ for each $x \in I_{n,m}$ and each i, and $I_{n,m} \subset \mathrm{C}(\alpha_{n-1})$, we can write the interval $I_{n,m}$ as the union of nonoverlapping compact intervals, $I_{n,m} = \bigcup_{j=1}^{s_{n,m}} I_{n,m,j}$, such that

$$\operatorname{diam} I_{n,m,j} < m^{-1} \varrho(I_{n,m}, A_n) \quad \text{and} \quad \omega(f_i + \alpha_{n-1}, I_{n,m,j}) < \tau_n \tag{1}$$

for all i and j.

Fix $m \in \mathbb{N}$ and $j \in \{1, \ldots, s_{n,m}\}$. Set

$$L_{n,m,j} = \begin{cases} \tau_{n-1} & \text{if } \tau_{n-1} < \infty\,, \\ \max\{\sup|f_i + \alpha_{n-1}|[I_{n,m,j}] \colon i \in \{1, \ldots, k\}\} + m & \text{otherwise}. \end{cases}$$

(The lower case is possible only if $n = 1$.) By Lemma 3.2, there exists a nowhere dense perfect set $F_{n,m,j} \subset I_{n,m,j} \cap C \setminus F_{n-1}$ and a continuous function $\alpha_{n,m,j}$ such that:

$$|\alpha_{n,m,j}| < L_{n,m,j} + \tau_n \ \text{ on } \mathbb{R}\,, \tag{2}$$

$$\alpha_{n,m,j} = 0 \ \text{ on } F_{n-1} \cup (\mathbb{R} \setminus I_{n,m,j})\,, \tag{3}$$

$$\int\limits_{I_{n,m,j}} |\alpha_{n,m,j}| < 2^{-m} \operatorname{diam} I_{n,m,j}/s_{n,m}\,, \tag{4}$$

for each i the set $(f_i + \alpha_{n-1} + \alpha_{n,m,j})[F_{n,m,j}]$ contains the interval

$$\Big[\inf(f_i + \alpha_{n-1})[I_{n,m,j}] - L_{n,m,j}\,,\ \sup(f_i + \alpha_{n-1})[I_{n,m,j}] + L_{n,m,j}\Big]\,. \tag{5}$$

Define

$$F_n = F_{n-1} \cup A_n \cup \bigcup_{m\in\mathbb{N}} \bigcup_{j=1}^{s_{n,m}} F_{n,m,j}\,, \qquad \alpha_n = \alpha_{n-1} + \sum_{m\in\mathbb{N}} \sum_{j=1}^{s_{n,m}} \alpha_{n,m,j}\,.$$

Clearly the set F_n is closed and nowhere dense. Using condition (4) one can easily show that each $x \in A_n$ is a Lebesgue point of $\alpha_n - \alpha_{n-1}$. Since α_n is continuous outside of A_n and α_{n-1} is a Lebesgue function, α_n is a Lebesgue function as well.

Put $\alpha = \lim\limits_{n\to\infty} \alpha_n$. By (2), this sequence is uniformly convergent, so α is a Lebesgue function. Moreover by construction, $\mathrm{C}(\alpha) \supset \bigcap\limits_{n\in\mathbb{N}} (\mathbb{R} \setminus A_n) = C$ and

$$|\alpha| < \tau_0 + 2\sum_{n\in\mathbb{N}} \tau_n = \sup\{\omega(f_i, x) \colon i \in \{1, \ldots, k\},\, x \in \mathbb{R}\} + \tau$$

on $\mathbb{R}$.

Fix an $i \in \{1, \ldots, k\}$. First we will show that

$$\text{for each } x \notin C \text{ and each } \mathscr{T}_{\mathcal{I}}\text{-open set } U \ni x\,, \text{ there is an open interval } I \sqsubset U \cap (x, \infty) \text{ with } I \cap C \cap (f_i + \alpha)^{-1}\big((f_i + \alpha)(x)\big) \neq \emptyset\,. \tag{6}$$

Let $x \notin C$ and let $U \ni x$ be an arbitrary $\mathscr{T}_{\mathcal{I}}$-open set. Let $n \in \mathbb{N}$ be such that $x \in A_n \setminus A_{n-1}$, and choose $\eta > 0$ and $r \in \mathbb{N}$ according to Lemma 3.1. We consider two cases.

CASE 1. If $\tau_{n-1} = \infty$, then pick a $\delta \in (0, \eta)$ such that

$$(x - \delta, x + \delta) \cap I_{n,m} = \emptyset \qquad \text{for each } m \leq |(f_i + \alpha)(x)| + 4r. \tag{7}$$

There is a $q \in \{1, \dots, r\}$ such that

$$I = \left(x + \frac{q-1}{r}\,\delta\,,\ x + \frac{q}{r}\,\delta\right) \sqsubset U\,.$$

Since the set A_n is nowhere dense and $\bigcup_{m\in\mathbb{N}} \bigcup_{j=1}^{s_{n,m}} I_{n,m,j} = \mathbb{R} \setminus A_n$, there are $m \in \mathbb{N}$ and $j \in \{1, \dots, s_{n,m}\}$ such that

$$I_{n,m,j} \cap \left(x + \frac{4q-3}{4r}\,\delta\,,\ x + \frac{4q-1}{4r}\,\delta\right) \neq \emptyset\,. \tag{8}$$

Then by (1) and (7), we obtain

$$\operatorname{diam} I_{n,m,j} < m^{-1} \varrho(I_{n,m}, A_n) < (4r)^{-1} \frac{4q-1}{4r}\,\delta < \frac{1}{4r}\,\delta\,,$$

whence

$$I_{n,m,j} \subset \left(x + \frac{4q-3}{4r}\,\delta - \operatorname{diam} I_{n,m,j}\,,\ x + \frac{4q-1}{4r}\,\delta + \operatorname{diam} I_{n,m,j}\right) \subset I\,.$$

By (7) and (8), $m > \big|(f_i + \alpha)(x)\big|$. So by (5), there is a $t \in F_{n,m,j} \subset I \cap C$ with $\big(f_i + \alpha_{n-1} + \alpha_{n,m,j}\big)(t) = (f_i + \alpha_{n-1})(x)$. But by (3), $(f_i + \alpha)(t) = \big(f_i + \alpha_{n-1} + \alpha_{n,m,j}\big)(t)$ and $(f_i + \alpha)(x) = (f_i + \alpha_{n-1})(x)$. This proves the claim (6) in this case.

CASE 2. If $\tau_{n-1} < \infty$, then $\omega(f_i + \alpha_{n-1}, x) < \tau_{n-1}$, and we can pick a $\delta > 0$ such that

$$\omega\big(f_i + \alpha_{n-1}, [x - \delta, x + \delta]\big) < \tau_{n-1}\,. \tag{9}$$

Without loss of generality we may assume that $\delta < \eta$ and

$$(x - \delta, x + \delta) \cap I_{n,m} = \emptyset \qquad \text{for each } m \leq 4r\,.$$

There is a $q \in \{1, \dots, r\}$ such that $I = \big(x + (q-1)\delta/r,\, x + q\delta/r\big) \sqsubset U$. Proceeding as above we can find $m \in \mathbb{N}$ and $j \in \{1, \dots, s_{n,m}\}$ such that $I_{n,m,j} \subset I$. By (9) and (5), there is a $t \in F_{n,m,j} \subset I \cap C$ with $\big(f_i + \alpha_{n-1} + \alpha_{n,m,j}\big)(t) = (f_i + \alpha_{n-1})(x)$. But by (3), $(f_i + \alpha)(t) = \big(f_i + \alpha_{n-1} + \alpha_{n,m,j}\big)(t)$ and $(f_i + \alpha)(x) = (f_i + \alpha_{n-1})(x)$. This proves claim (6).

Let $x \notin C$ and $\eta > 0$. Then by (6), there is an open interval $I \sqsubset (x, x + \eta)$ with $I \cap C \cap (f_i + \alpha)^{-1}\big((f_i + \alpha)(x)\big) \neq \emptyset$. One can easily see that $I \subset (x, x + \eta)$, and consequently

$$(f_i + \alpha)(x) \in (f_i + \alpha)[I \cap C] \subset (f_i + \alpha)\big[(x, x + \eta) \cap C\big]\,.$$

Proceeding analogously we can show that $(f_i+\alpha)(x) \in (f_i+\alpha)\big[(x-\eta, x)\cap C\big]$ for each $x \notin C$ and each $\eta > 0$. Consequently, $(f_i+\alpha)[I] \subset (f_i+\alpha)[I\cap C]$ for each closed interval I and the function $f_i+\alpha$ is strong Świątkowski.

Finally we will prove that the function $f_i+\alpha$ is $\mathscr{T}_{\mathcal{I}}$-quasi-continuous. If $x \in C$, then $x \in \mathrm{C}(f_i+\alpha)$. So let $x \notin C$.

Fix an $\varepsilon > 0$ and a $\mathscr{T}_{\mathcal{I}}$-open set $U \ni x$. By (6), there is an open interval $I \sqsubset U$ with $I\cap C\cap (f_i+\alpha)^{-1}\big((f_i+\alpha)(x)\big) \neq \emptyset$. There is an open interval $J \subset I$ such that

$$\big|(f_i+\alpha)(t) - (f_i+\alpha)(x)\big| < \varepsilon/3 \qquad \text{for each } t \in J\,.$$

Since $I \sqsubset U$, we have $V = J\cap U \neq \emptyset$. Clearly the set V is $\mathscr{T}_{\mathcal{I}}$-open and

$$\omega\big(f,\, \{x\}\cup V\big) \le \omega\big(f,\, \{x\}\cup J\big) < \varepsilon\,.$$

This completes the proof. □

The next corollary is a generalization of [8, Theorem 4.2].

COROLLARY 4.2. *Given a cliquish function f, we can find a $\mathscr{T}_{\mathcal{I}}$-quasi-continuous strong Świątkowski Lebesgue function α such that $f-\alpha$ is strong Świątkowski and $\mathscr{T}_{\mathcal{I}}$-quasi-continuous, and $\mathrm{C}(\alpha) \supset \mathrm{C}(f)$. Moreover, we can require that α is bounded provided that f is bounded.*

P r o o f. Use Theorem 4.1 for the family $\{-f, 0\}$. □

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