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ON DARBOUX $\mathbb{Q}$-DIFFERENTIABLE FUNCTIONS AND DARBOUX WRIGHT CONVEX FUNCTION

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ABSTRACT. It is shown that for each function $f : \mathbb{R} \to \mathbb{R}$ there is an additive almost continuous function g (an additive function h) such that the sum $f+g$ $(f+h)$ is almost continuous in the sense of Stallings (does not have the Darboux property).

Let $\mathbb{R}$ be the set of all reals.

Additive functions play a very important role in the theory of functional equations and inequalities.

A function $h\colon \mathbb{R} \to \mathbb{R}$ is said to be additive if it satisfies Cauchy's equation

$$h(x+y) = h(x) + h(y) \qquad \text{for } x, y \in \mathbb{R}\,.$$

An additive function $h\colon \mathbb{R} \to \mathbb{R}$ is continuous if and only if there is a real a with $h(x) = ax$ for $x \in \mathbb{R}$ ([5]).

A function $g\colon \mathbb{R} \to \mathbb{R}$ is said almost continuous (in the sense of Stallings) if for each open set $U \subset \mathbb{R}^2$ containing the graph $\mathrm{Gr}(g)$ of the function g there is a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ with $\mathrm{Gr}(f) \subset U$ ([6]).

Evidently each almost continuous function has the Darboux property.

In [3,4] some examples of discontinuous additive almost continuous functions from $\mathbb{R}$ to $\mathbb{R}$ are shown.

Additive functions appear in different divisions of theory of functional equations and inequalities. For example, a function $f\colon \mathbb{R} \to \mathbb{R}$ is a Wright-convex function if and only if it is the sum of an additive function and a convex function (see [7]). Moreover, a function $g\colon \mathbb{R} \to \mathbb{R}$ is strongly $\mathbb{Q}$-differentiable if and only if it is the sum of an additive function and a continuously differentiable function (see [1]).

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In this article we show some general theorems, which imply the answers to the questions if there are any Wright-convex functions and strongly $\mathbb{Q}$-differentiable functions with (without) the Darboux property.

For this, we remember that a closed set $K \subset \mathbb{R}^2$ is said to be a blocking set for a function $f\colon \mathbb{R} \to \mathbb{R}$ if $K \cap \mathrm{Gr}(f) = \emptyset$ and $K \cap \mathrm{Gr}(h) \neq \emptyset$ for every continuous function $h\colon \mathbb{R} \to \mathbb{R}$. An irreducible blocking set (IBS) K of f is a blocking set of f such that no proper subset of K is a blocking set. It is known that f is an almost continuous function if and only if it has no blocking set. Moreover, if f is not almost continuous, then there is an (IBS) K of f, and the x-projection $\mathrm{pr}_x(K)$ is a nondegenerate connected set (see [4]).

Throughout the article we assume Continuum Hypothesis **CH**. Let ω_1 be the first uncountable ordinal number and let

$$K_0, K_1, \dots, K_\alpha, \dots; \quad \alpha < \omega_1\,, \tag{$*$}$$

be a transfinite sequence of all irreducible blocking sets in $\mathbb{R}^2$ of functions from $\mathbb{R}$ to $\mathbb{R}$.

Theorem 1. *For each function $f\colon \mathbb{R} \to \mathbb{R}$ there is an almost continuous additive function $g\colon \mathbb{R} \to \mathbb{R}$ such that the function $f+g$ is almost continuous.*

P r o o f. Let $H \subset \mathbb{R}$ be a Hamel basis (of the linear space $\mathbb{R}$ over the field $\mathbb{Q}$ of all rationals) such that for each open interval I the intersection $I \cap H$ is of the cardinality continuum (see [5, 2]).

By transfinite induction, in each projection $\mathrm{pr}_x(K_\alpha)$, where $\alpha < \omega_1$, we find two points $x_{1,\alpha}, x_{2,\alpha} \in \mathrm{pr}_x(K_\alpha) \cap H$ such that

$$x_{i,\alpha} \neq x_{j,\beta} \qquad \text{if} \quad (i,\alpha) \neq (j,\beta)\,, \quad i,j = 1,2 \ \text{and} \ \alpha\,,\beta < \omega_1\,.$$

Moreover, for each ordinal $\alpha < \omega_1$ we find two real $y_{1,\alpha}$ and $y_{2,\alpha}$ with

$$(x_{1,\alpha}\,, y_{1,\alpha}) \in K_\alpha \quad \text{and} \quad \big(x_{2,\alpha}\,,\ y_{2,\alpha} + f(x_{2,\alpha})\big) \in K_\alpha\,.$$

Let

$$\phi(x) = \begin{cases} y_{1,\alpha} & \text{if } \ x = x_{1,\alpha}\,,\ \alpha < \omega_1\,, \\ y_{2,\alpha} & \text{if } \ x = x_{2,\alpha}\,,\ \alpha < \omega_1\,, \\ 0 & \text{otherwise on } H \end{cases}$$

and let $g\colon \mathbb{R} \to \mathbb{R}$ be the additive extension of the function $\phi\colon H \to \mathbb{R}$. Then g is an additive function. Since for each ordinal $\alpha < \omega_1$ we have

$$\mathrm{Gr}(g) \cap K_\alpha \neq \emptyset \quad \text{and} \quad \mathrm{Gr}(f+g) \cap K_\alpha \neq \emptyset\,,$$

both the functions g and $f+g$ are almost continuous. □

THEOREM 2. *For each function $f\colon \mathbb{R} \to \mathbb{R}$ there is an additive function $g\colon \mathbb{R} \to \mathbb{R}$ such that the sum $f + g$ does not have the Darboux property.*

P r o o f . There is a Hamel basis $H \subset (0,1) \setminus \mathbb{Q}$ (see [5]). Let

$$H = \{h_\alpha;\ 0 \le \alpha < \omega_1\}, \qquad \text{where } h_\alpha \ne h_\beta \text{ for } \alpha \ne \beta,$$

and for each ordinal $\alpha < \omega_1$ let H_α be the linear space over $\mathbb{Q}$ generated by the set

$$Z_\alpha = \{h_\beta \colon \beta \le \alpha\}.$$

Observe that each set H_α, $\alpha < \omega_1$, is countable,

$$H_\alpha \subset H_\beta \qquad \text{for} \quad \alpha < \beta < \omega_1,$$

and

$$\mathbb{R} = \bigcup_{\alpha<\omega_1} H_\alpha .$$

Now, we will define, by transfinite induction, an additive function $g\colon \mathbb{R} \to \mathbb{R}$ satisfying all the requirements.

For this we find values $\phi(h_0)$ and $\phi(h_1)$ such that

$$\phi(h_0) + f(h_0) < 0 \quad \text{and} \quad \phi(h_1) + f(h_1) > 0$$

and for all pairs $(r,s) \ne (0,0)$ of rationals we have

$$r\phi(h_0) + s\phi(h_1) \ne -f(rh_0 + sh_1).$$

Such values $\phi(h_0)$ and $\phi(h_1)$ exist because the set H_1 and the set of all pairs of rationals are countable.

Moreover, for $x \in H_1$ choose $r, s \in \mathbb{Q}$ such that $x = rh_0 + sh_1$ and let

$$g_1(x) = r\phi(h_0) + s\phi(h_1).$$

Since the rational coefficients in this representation are unique, we have defined a function $g_1\colon H_1 \to \mathbb{R}$. It is clear that g_1 is additive and

$$g_1(x) + f(x) \ne 0 \qquad \text{for every } x \in H_1 \setminus \{0\}.$$

Fix a countable ordinal $\alpha > 1$ and suppose that we have defined, for every $\beta < \alpha$, an additive function $g_\beta\colon H_\beta \to \mathbb{R}$ such that

$$g_\beta(x) + f(x) \ne 0 \qquad \text{for every } x \in H_\beta \setminus \{0\}$$

and g_γ is the restriction of g_β to H_γ if $\gamma < \beta < \alpha$.

Consider the set H_α. For each point $x \ne 0$ belonging to H_α there exists exactly one pair (u, r), where $r \in \mathbb{Q}$ and $u \in H_\beta$, with $\beta < \alpha$, such that $x = u + rh_\alpha$.

Since for each rational $r \neq 0$ there is only one real $y_{r,u}$ such that

$$g_\beta(u) + r y_{r,u} + f(u + r h_\alpha) = 0 \tag{1}$$

and since the sets $\mathbb{Q}$ and

$$W_\alpha = \bigcup_{\beta<\alpha} H_\beta$$

are countable, there is a real

$$\phi(h_\alpha) \neq y_{r,u} \qquad \text{for all pairs } (r,u) \in \mathbb{Q} \times W_\alpha\,. \tag{2}$$

Now, for $x \in H_\alpha$ choose $\beta < \alpha$, $u \in H_\beta$ and $r \in \mathbb{Q}$ such that $x = u + rh_\alpha$ and let $\bigl(g_\alpha(x) = g_\beta(u) + r\phi(h_\alpha)\bigr)$. Then $g_\alpha \colon H_\alpha \to \mathbb{R}$ is additive,

$$g_\alpha(x) + f(x) \neq 0 \qquad \text{for every } x \in H_\alpha \setminus \{0\}\,,$$

and, for every $\beta < \alpha$, g_β is the restriction of g_α to H_β. Thus,

$$g = \bigcup_{\alpha<\omega_1} g_\alpha$$

is an additive function from $\mathbb{R}$ to $\mathbb{R}$ such that

$$g(x) + f(x) \neq 0 \qquad \text{for every } x \in \mathbb{R} \setminus \{0\}\,. \tag{3}$$

Note that $h_0, h_1 \in (0,1)$,

$$f(h_0) + g(h_0) = f(h_0) + \phi(h_0) < 0, \quad \text{and} \quad f(h_1) + g(h_1) = f(h_1) + \phi(h_1) > 0\,.$$

Moreover, if $x \in (0,1)$ then $f(x) + g(x) \neq 0$ by (3).

So, the sum $f + g$ does not have the Darboux property and the proof is finished. □

However, if $f(x) = ax$ for $x \in \mathbb{R}$, then for each additive almost continuous function $g \colon \mathbb{R} \to \mathbb{R}$ the sum $f + g$ is almost continuous (so it has the Darboux property) ([2]).

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