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SEMI-QUASICONTINUITY AND PU–CHEN–PU THEOREM

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ABSTRACT. In connection with a widening of the Pu–Chen–Pu Theorem the four classes of semi-quasicontinuous functions are studied.

We begin with stating the Theorem of Pu–Chen–Pu (see [8]).

Theorem. *Assume the function* $f\colon \mathbb{R} \to \mathbb{R}$ *is continuous. Then*

$$\{x\colon AD^- f(x) < D^+ f(x) \ \text{ or } \ AD^+ f(x) < D^- f(x)\}$$

is of the first category.

In the above Theorem, $AD^- f(x)$, $AD^+ f(x)$, $D^- f(x)$ and $D^+ f(x)$ denote, respectively, upper left and right approximate derivate and Dini derivative. The original proof of this Theorem was given in [8], however, the proof presented in [9] (we will present its outline later) is shorter and can be applied to a wider class of functions. In [9], the authors use the following properties of continuous functions.

$$\begin{aligned}
&(\mathrm{LUSC}) \quad \forall_{x,y}\big(f(x) < y \Rightarrow \exists_{x'} \forall_{x'' \in (x',x)} f(x'') < y\big),\\
&(\mathrm{LLSC}) \quad \forall_{x,y}\big(f(x) > y \Rightarrow \exists_{x'} \forall_{x'' \in (x',x)} f(x'') > y\big),\\
&(\mathrm{RUSC}) \quad \forall_{x,y}\big(f(x) < y \Rightarrow \exists_{x'} \forall_{x'' \in (x,x')} f(x'') < y\big).\\
&(\mathrm{RLSC}) \quad \forall_{x,y}\big(f(x) > y \Rightarrow \exists_{x'} \forall_{x'' \in (x,x')} f(x'') > y\big),
\end{aligned}$$

The letters LU, LL, RU and RL appearing in the above definitions stand, respectively, for "left upper", "left lower", "right upper" and "right lower". SC stands for "semicontinuity". Indeed, LUSC and LLSC imply the left semicontinuity and LUSC together with RUSC implies upper semicontinuity.

The four semicontinuity properties, i.e., LUSC, LLSC, RUSC and RLSC, are not satisfied by $\mathcal{DB}_1^*$ functions, for one can take the function $f\colon \mathbb{R} \to \mathbb{R}$ given

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by formulas $f(x)=\sin\left(-\frac{1}{x}\right)$ for $x\neq 0$ and $f(0)=0$. Actually, the fact that a function satisfies the four semicontinuity properties is equivalent to continuity. We recall that a function $f\colon A\to\mathbb{R}$ is of Baire class one star ($\mathcal{B}_1^*$), if for any nonempty closed set B, there is an open interval (a,b) such that $(a,b)\cap A\cap B\neq\emptyset$ and $f|_{A\cap B}$ is continuous on $(a,b)\cap A\cap B$. The $\mathcal{DB}_1^*$ functions, i.e., Darboux functions of Baire class one star, were studied by O'Malley in [7] and Kirchheim in [4].

We introduce conditions that are satisfied by $\mathcal{DB}_1^*$ functions.

$$\text{(LUSQC)}\quad \forall_{x,y}\big(f(x)<y\Rightarrow\forall_{x_1<x}\,\exists_{x_1<x_1'<x'<x}\,\forall_{x''\in(x_1',x')}\,f(x'')<y\big)\,,$$

$$\text{(LLSQC)}\quad \forall_{x,y}\big(f(x)>y\Rightarrow\forall_{x_1<x}\,\exists_{x_1<x_1'<x'<x}\,\forall_{x''\in(x_1',x')}\,f(x'')>y\big)\,,$$

$$\text{(RUSQC)}\quad \forall_{x,y}\big(f(x)<y\Rightarrow\forall_{x_1>x}\,\exists_{x_1>x_1'>x'>x}\,\forall_{x''\in(x',x_1')}\,f(x'')<y\big)\,.$$

$$\text{(RLSQC)}\quad \forall_{x,y}\big(f(x)>y\Rightarrow\forall_{x_1>x}\,\exists_{x_1>x_1'>x'>x}\,\forall_{x''\in(x',x_1')}\,f(x'')>y\big)\,,$$

Like above, LU, LL, RU and RL stand, respectively, for "left upper", "left lower", "right upper" and "right lower" while SQC stands for "semi-quasicontinuity". We say that a *function satisfies all four semi-quasicontinuity properties* if it satisfies LUSQC, LLSQC, RUSQC and RLSQC. We may also introduce the local notion of the above properties. For example, we say that $f\colon A\to\mathbb{R}$ satisfies the LUSQC property at the fixed point x_0 if for any real y, the inequality $f(x_0)<y$ implies that for each $x_1<x_0$ there exists a subinterval $(x_1',x_0')\subset(x_1,x_0)$ on which f takes only the values strictly less than y. It is clear that a function quasicontinuous at the fixed point x_0 satisfies at x_0 at least two of the above four semi-quasicontinuity properties.

Proposition 1. *Any function $f\in\mathcal{DB}_1^*$ satisfies all four semi-quasicontinuity properties.*

Proof. We shall prove that f has the property LLSQC as the over three proofs are quite similar. Assume for some arbitrary x and y that $f(x)>y$. If x is the point of continuity of f then there is nothing to prove. Otherwise, take $x_1<x$ and denote by U the interior of the set of continuity points of f. We have $\{z\colon f(z)>y\}\cap[x_1,x]\neq\emptyset$. It is shown in [7] that $\{z\colon f(z)>y\}\cap[x_1,x]\cap U\neq\emptyset$. In particular, f has in (x_1,x) a continuity point x' with $f(x')>y$. Thus, in (x_1,x') there exists x_1' such that $f(x'')>y$ for all $x''\in(x_1',x')$. □

Remark. The property LLSQC need not be satisfied by a Baire class one Darboux function. As an example we consider the Croft function $F\colon[0,1]\to\mathbb{R}$ (see [2], page 11). This function takes value 0 on the dense set $E=\left\{\frac{k}{2^n}\colon n\in\mathbb{N}, 1\leq k\leq 2^n-1\right\}$. However, $F\left(\frac{1}{3}\right)>0$. Thus, for any x_1, there is no open subinterval (x_1',x') of $\left[x_1,\frac{1}{3}\right]$ on which F is positive.

Proof of Pu–Chen–Pu Theorem. By using Proposition 1, we shall prove the following version of the Theorem. □

Theorem. *Assume the function $f\colon \mathbb{R} \to \mathbb{R}$ is a $\mathcal{DB}_1^*$ function. Then*

$$\{x\colon AD^- f(x) < D^+ f(x) \ \text{ or } \ AD^+ f(x) < D^- f(x)\}$$

is of the first category.

Proof. Roughly we follow the lines of the proof presented in [9]. It suffices to show that the set $A = \{x\colon AD^- f(x) < D^+ f(x)\}$ is of the first category. To do this, we notice that $A \subset \bigcup_{n\in\mathbb{N}} \bigcup_{r,s\in\mathbb{Q}} A_{r,s,n}$, where $A_{r,s,n}$ is the set of those $x \in \mathbb{R}$ for which $D^+ f(x) > s > r > AD^- f(x)$ and

$$m\left\{y \in (x-h, x)\colon \frac{f(x)-f(y)}{x-y} > r\right\} < \frac{h}{3} \quad \text{for } h \in \left(0, \frac{1}{n}\right). \tag{1}$$

Here m denotes the Lebesgue measure on $\mathbb{R}$. It is enough to show that given $r, s \in \mathbb{Q}$ with $r < s$ and $n \in \mathbb{N}$ the set $A_{r,s,n}$ is nowhere dense. Suppose it is not the case, i.e., $A_{r,s,n}$ is dense in some interval (α, β) of the length less than $1/n$. As $f \in \mathcal{DB}_1^*$, the inside of the set of continuity points of f must be dense in (α, β). As $A_{r,s,n}$ is also dense, there exists a point $x_0 \in A_{r,s,n} \cap (\alpha, \beta)$ which is a continuity point of f. Take any $x_1 \in (x_0, \beta)$ with

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} > s\,.$$

Set $L(x) = f(x_0) + s(x_1 - x_0) + r(x - x_1)$ and see that by the above, $f(x_1) - L(x_1) > 0$. Using the LLSQC we get the existence of a subinterval (x_0', x_1') of (x_0, x_1) on which $f - L$ is positive. As $(x_0', x_1') \subset (\alpha, \beta)$, there is $\overline{x}_1 \in (x_0', x_1') \cap A_{r,s,n}$ with $f(\overline{x}_1) > L(\overline{x}_1)$. Further on, we have $\overline{x}_1 - x_0 < \frac{1}{n}$, so there is $x_2 \in (x_0, \overline{x}_1)$ with

$$x_2 - x_0 < \tfrac{1}{3}(\overline{x}_1 - x_0) \quad \text{and} \quad \frac{f(x_2) - f(\overline{x}_1)}{x_2 - \overline{x}_1} \le r\,.$$

As $x_2 - \overline{x}_1 < 0$, we get

$$L(x_2) = L(\overline{x}_1) + L(x_2) - L(\overline{x}_1) < f(\overline{x}_1) + r(x_2 - \overline{x}_1) \le f(x_2)\,.$$

Similarly, like above, we find $\overline{x}_2 \in A_{r,s,n} \cap (x_0, x_2)$ such that $f(\overline{x}_2) > L(\overline{x}_2)$. Continuing the process, one obtains a decreasing sequence $(\overline{x}_n)$ of elements of $A_{r,s,n}$ that converges to x_0 and $f(\overline{x}_n) > L(\overline{x}_n)$ for all n.

Now, we readily see that $\overline{x}_n - x_0 < \frac{1}{n}$ and that the condition

$$\frac{f(\overline{x}_n) - f(x)}{\overline{x}_n - x} \le r$$

implies $f(x) > L(x)$. Therefore by (1),

$$m\{x \in (x_0, \overline{x}_n) \colon f(x) > L(x)\} \geq \tfrac{2}{3}(\overline{x}_n - x_0),$$

and the last implies

$$\limsup_{x \to x_0^+} f(x) \geq \limsup_{x \to x_0^+} L(x) = L(x_0) > f(x_0),$$

which contradicts the continuity of f at x_0. The proof of the theorem is complete. $\square$

Semi-quasicontinuous functions and the properties of Baire and Darboux

We shall study the functions that have all four semi-quasicontinuity properties.

Proposition 2. *If a Baire class one functions satisfies all four semi-quasicontinuity properties, then it is also the Darboux function.*

Proof. Assume $a < b$ and $f(a) > f(b)$. Let $y \in \big(f(b), f(a)\big)$. Suppose f does not have the Darboux property and it does not achieve value y on $[a, b]$. Define

$$U = \operatorname{int}\{x \in [a, b] \colon f(x) > y\} \quad \text{and} \quad L = \operatorname{int}\{x \in [a, b] \colon f(x) < y\}.$$

By RLSQC and LUSQC, the sets U and L are nonempty. We write U and L as disjoint sums of intervals $U = \bigcup_i (u_i, u_i')$ and $L = \bigcup_j (v_j, v_j')$. If for some i one of the inequalities $f(u_i) < y$ or $f(u_i') < y$ holds, then we have an immediate contradiction with RUSQC or LUSQC. For similar reasons it cannot happen that $f(v_j) > y$ or $f(v_j') > y$ for some j.

Now we consider the case when $f(x) > y$ for any $x \in \bigcup_i [u_i, u_i']$ and $f(x) < y$ for $x \in \bigcup_j [v_j, v_j']$. One can notice that the above sums are disjoint. Thus the set $K = [a, b] \setminus (U \cup L)$ is perfect and its interior is empty. We shall show that $f|_K$ has no continuity points contradicting the fact that $f \in \mathcal{B}_1$. Let $x \in K$. We may assume that $f(x) > y$. Consider the sets $A = \bigcup_i \{u_i, u_i'\}$ and $B = \bigcup_j \{v_j, v_j'\}$. By RLSQC and LLSQC, there exist $a \in A$ in each neighborhood of x, so A is dense in K. Now, if $x \in A$, and x is equal to u_i, then in each interval (x_1, x) for $x_1 < x$ there must be an element from B. Similarly, if $x = u_i'$ for some i, there is an element from B in each right hand side neighborhood of x. If $x \notin A$, then in each neighborhood (x_1, x_2) of x one can find an element $a \in A$, so, by the above, in (x_1, x_2) there must be an element from B. Thus B is dense in K. Since A and B are dense in K and $f(a) > y$ for $a \in A$ and $f(b) < y$ for $b \in B$, any point of K is a discontinuity point of $f|_K$.

We get to the same conclusion if we assume $f(a) < f(b)$. The proof is complete. □

The following example shows that the assumption $f \in \mathcal{B}_1$ is crucial.

EXAMPLE 3. Let K denote the Cantor-like set with the only exception that instead of throwing out open intervals, on each step of construction we throw out closed intervals of lengths 3^{-n}. We define $f(x) = 1$ for $x \in K$ or x belonging to the contiguous intervals of length 3^{-2k} and $f(x) = -1$ for x belonging to the thrown out intervals of length 3^{-2k-1}, where k is a positive integer and $f\colon [0,1] \to \mathbb{R}$. So, defined function f has all four semi-quasicontinuity properties, however, it does not have the Darboux property.

The $\mathcal{DB}_1$ function having all four semi-quasicontinuity properties does not need to be $\mathcal{DB}_1^*$. Examples of such functions could be found in [6]. Here we give another one.

EXAMPLE 4. Consider the sequence (q_n) of all rational numbers in $[0,1]$. Let $f_n\colon [0,1] \to \mathbb{R}$ be given by formula

$$f_n(x) = \begin{cases} 2^{-n} \sin \frac{1}{x-q_n} & \text{if } x \neq q_n\,; \\ 0 & \text{otherwise}\,. \end{cases}$$

We define $f(x) = \sum\limits_{n=1}^{\infty} f_n(x)$. The function f is of Baire class one as a uniform limit of such functions. Also by uniform convergence, the function f is continuous at each irrational point. Moreover, because the functions f_n are bilaterally quasicontinuous, their uniform limit is bilaterally quasicontinuous (see [1]). Also, as is shown in [1], $\mathcal{B}_1$ bilaterally quasicontinuous function is Darboux. Therefore $f \in \mathcal{DB}_1$ and it satisfies all four semi-quasicontinuity properties. Since the set of discontinuity points of f is dense, it cannot be $\mathcal{B}_1^*$.

Further properties of semi-quasicontinuous functions

We finish this paper with the following

PROPOSITION 5. *The set where some function f is* LLSQC, *but it is not* RLSQC *is at most countable.*

P r o o f. Assume A is the set where f is LLSQC, but not RLSQC. Let $a \in A$. As f is not RLSQC at a, we have

$$\exists_{y_a} \big(f(a) > y_a \text{ and } \exists_{a'>a} \forall_{(a_1,a_2) \subset (a,a')} \exists_{x \in (a_1,a_2)} f(x) \leq y_a \big)\,.$$

We consider the mapping

$$\Phi\colon a \mapsto (a, a') \times \big(y_a, f(a)\big)$$

for $a \in A$. Suppose $a < b$ and $a, b \in A$. We shall show $\Phi(a) \cap \Phi(b) = \emptyset$. Assume it is not the case, i.e.,

$$(a, a') \cap (b, b') \neq \emptyset$$

and

$$\big(y_a, f(a)\big) \cap \big(y_b, f(b)\big) \neq \emptyset .$$

In particular, $b < a'$ and $f(b) > y_a$. However, the function f is LLSQC at b, so, there is an interval $(x_1, x_2) \subset (a, b)$ such that

$$\forall_{x \in (x_1, x_2)} f(x) > y_a .$$

But this contradicts the choice of y_a or a' as $(x_1, x_2) \subset (a, a')$ and $f(x) > y_a$ for $x \in (x_1, x_2)$.

Thus we have that Φ is 1–1 and $\{\Phi(a)\}_{a \in A}$ is a collection of disjoint open squares on the plane. This implies that A is at most countable. $\square$

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