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STRONG SUPMEASURABILITY OF FUNCTIONS OF TWO VARIABLES

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ABSTRACT. We introduce and investigate the strong supmeasurability of functions of two variables.

Let $\mathbb{R}$ and $\mathbb{N}$ denote the real line and the set of positive integers, respectively. Let $(X, \mathscr{M})$ be a measurable space and let $\mathcal{I} \subset \mathscr{M}$ be a proper σ-ideal of subsets of X. Let Z be a Banach space.

Let $h\colon X \to Z$. Recall that h is *measurable*, if $h^{-1}(U) \in \mathscr{M}$ for every open set $U \subset Z$. In [5] the following were introduced two kinds of measurability of a function. We say that h is *nearly simple*, if it is measurable and its range is at most countable. We say that h is *strongly measurable* with respect to $(\mathscr{M}, \mathcal{I})$, if there exists a sequence of nearly simple functions (h_n) and a set $A \in \mathcal{I}$ such that $h_n \to h$ on $X \setminus A$.

Clearly, Bochner integrable functions are strongly measurable, because they are limits of simple functions (i.e., measurable functions with finite range). Moreover, strongly measurable functions are measurable. Easy examples show that the converse implications do not hold.

Remark 1. In functional analysis (see, e.g., [19]) we usually consider another kind of strong measurability, namely, we require that there exists a sequence of simple functions (h_n) and a set $A \in \mathcal{I}$ such that $h_n \to h$ on $X \setminus A$. If Z is a separable (in particular, finitely dimensional) Banach space, then this notion is equivalent to the measurability, and consequently, to the strong measurability defined above. However, we do not know whether two notions of strong measurability coincide in a general case.

Problem. Assume that $f\colon X \to Z$ is strongly measurable. Does there exist a sequence of simple functions $f_n\colon X \to Z$ such that $f_n(x) \to f(x)$ for $\mathcal{I}$-almost every $x \in X$?

The following proposition is closely related to this problem.

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Proposition 1. *Assume that there is a σ-finite measure $\mu\colon \mathscr{M} \to [0,\infty]$ such that $\mathcal{I}$ is the σ-ideal of μ-null sets and $h\colon X \to Z$ is strongly measurable. There exists a sequence of simple functions $h_m\colon X \to Z$ such that $h_m(x) \to h(x)$ for $\mathcal{I}$-almost every $x \in X$.*

Proof. First assume that $\mu(X) < \infty$. Let (f_k) be a sequence of nearly simple functions which is μ-a.e. convergent to h. Fix an $m \in \mathbb{N}$.

For each $l \in \mathbb{N}$ define

$$A_{m,l} = \bigl\{x \in X\colon \|f_k(x) - h(x)\| \geq m^{-1} \text{ for some } k \geq l\bigr\}.$$

Since the sequence $(A_{m,l})_{l=1}^{\infty}$ is descending and $\mu\Bigl(\bigcap\limits_{l\in\mathbb{N}} A_{m,l}\Bigr) = 0$, there is an $L_m \in \mathbb{N}$ with $\mu(A_{m,L_m}) < m^{-1}$.

Choose $z_{m,1}, z_{m,2}, \dots \in Z$ and pairwise disjoint sets $X_{m,1}, X_{m,2}, \dots \in \mathscr{M}$ such that $f_{L_m} = z_{m,n}$ on $X_{m,n}$ for each n and $X = \bigcup\limits_{n\in\mathbb{N}} X_{m,n}$. Since $\mu(X) < \infty$, there is an $N_m \in \mathbb{N}$ with $\mu\Bigl(\bigcup\limits_{n>N_m} X_{m,n}\Bigr) < m^{-1}$. Define

$$g_m = \sum_{n\leq N_m} z_{m,n}\chi_{m,n}\,.$$

Clearly, g_m is a simple function.

Observe that

$$\begin{aligned}
&\mu\bigl(\{x \in X\colon \|g_m(x) - h(x)\| \geq m^{-1}\}\bigr)\\
&\leq \mu\bigl(\{x \in X\colon g_m(x) \neq f_{L_m}(x)\}\bigr) + \mu\bigl(\{x \in X\colon \|f_{L_m}(x) - h(x)\| \geq m^{-1}\}\bigr)\\
&\leq \mu\Bigl(\bigcup_{n>N_m} X_{m,n}\Bigr) + \mu\bigl(A_{m,L_m}\bigr) < 2m^{-1} \overset{m\to\infty}{\longrightarrow} 0\,.
\end{aligned}$$

So, the sequence (g_m) is convergent in measure to h. Consequently, we can choose a subsequence of this sequence convergent μ-a.e. to h.

Now we consider a general case. Let (X_n) be a sequence of pairwise disjoint $\mathscr{M}$-measurable sets of finite measure whose union is X. For each n, by the first part of the proof, there is a sequence $(h_{n,m})$ composed of simple functions defined on X_n, μ-a.e. convergent to $h\restriction X_n$. Define

$$h_m(x) = \begin{cases} h_{n,m}(x) & \text{if } x \in X_n\,,\ n \leq m\,,\\ 0 & \text{otherwise}\,. \end{cases}$$

One can easily see that the sequence (h_m) has all desired properties. □

We will need the following theorem containing a condition equivalent to the strong measurability. It is a consequence of [5, Corollary 3 and Lemma 4] and [6, Proposition 1].

Theorem 2. *Let $h\colon X \to Z$. The following are equivalent:*

i) *h is strongly measurable,*

ii) *h is measurable and*

$$\text{there is an } A \in \mathcal{I} \text{ such that } h(X \setminus A) \text{ is a separable subspace of } Z\,. \tag{1}$$

Moreover, if the condition ii) holds, then there exists a sequence of nearly simple functions convergent uniformly to h on $X \setminus A$.

Let $(Y, \mathscr{N}, \nu)$ be a measure space. We assume that we have a net structure $\mathscr{J}$ in $(Y, \mathscr{N}, \nu)$ (see, e.g., [3]). Recall that a *net* in Y is an at most countable cover of Y consisting of pairwise disjoint measurable sets of positive finite measure. The individual sets in the net are called *cells*. The family $\mathscr{J} = \bigcup_{n=1}^{\infty} \mathscr{J}_n$, where each $\mathscr{J}_n$ is a net, is called a *net structure*. Observe that for each $y \in Y$ and $n \in \mathbb{N}$, there is a unique cell from the net $\mathscr{J}_n$ which contains y. We denote this cell by $J_n(y)$. Several examples of net structures can be found, e.g., in [15] or [16].

In most theorems, functions from the Cartesian product $X \times Y$ into Z are considered. We will consider the product σ-ideal $\widetilde{\mathcal{I}}$ in $X \times Y$ (the family of all subsets of the sets of the form $A \times Y$, where $A \in \mathcal{I}$) and the product σ-field $\mathcal{M} \otimes \mathscr{N}$ (the smallest σ-field containing the σ-ideal $\widetilde{\mathcal{I}}$ and the family of all sets of the form $M \times N$, where $M \in \mathscr{M}$ and $N \in \mathscr{N}$). The strong measurability of a function $h\colon X \times Y \to Z$ can be defined analogously as in the above case; i.e., a function $f\colon X \times Y \to Z$ is strongly measurable, if there exists a sequence of nearly simple functions (f_n) from $X \times Y$ into Z and a set $A \in \mathcal{I}$ such that $f_n \to f$ on $(X \setminus A) \times Y$.

In the main theorems, we assume that the vertical sections of a given function are derivatives with respect to the net structure $\mathscr{J}$. Recall the appropriate definitions.

Let $f\colon X \times Y \to Z$. For each $x \in X$, the function $f_x\colon Y \to Z$ defined by $f_x(y) = f(x,y)$ is called a *vertical section* of f. Similarly, for each $y \in Y$, the function $f^y\colon X \to Z$ defined by $f^y(x) = f(x,y)$ is called a *horizontal section* of f.

A function $g\colon Y \to Z$ is called a *derivative* (with respect to the net structure $\mathscr{J}$) if g is Bochner integrable over each $J \in \mathscr{J}$, and for each $y \in Y$, we have

$$\lim_{n\to\infty} \frac{\int_{J_n(y)} g(v)\,\nu(dv)}{\nu\big(J_n(y)\big)} = g(y)\,.$$

The notion of supmeasurability of functions from the Cartesian product $X \times Y$ into Z needs some kind of measurability of functions from X into Y. In general, we have no topological structure in Y. Therefore we will use the net structure. One can easily see that for real functions defined on $\mathbb{R}$, the notion of supmeasurability defined below coincides with the classical one (see, e.g., [18]).

A function $\varphi\colon X \to Y$ is called $(\mathscr{M}, \mathscr{J})$-*measurable*, if for each $J \in \mathscr{J}$, the inverse image $\varphi^{-1}(J) \in \mathscr{M}$.

A function $f\colon X \times Y \to Z$ is *strongly supmeasurable* (i.e., superpositionally measurable), if for each $(\mathscr{M}, \mathscr{J})$-measurable function $\varphi\colon X \to Y$, the superposition $x \mapsto f\big(x, \varphi(x)\big)$ is strongly measurable.

Remark 2. Clearly, a function which is the pointwise limit of a sequence of strongly supmeasurable functions is strongly supmeasurable.

The notion of supmeasurability was investigated by many mathematicians. Most often a function from $\mathbb{R}^2$ into $\mathbb{R}$ is considered, and it is assumed that the σ-field in the definition of this notion is either the σ-field of all Lebesgue measurable sets or the σ-field of all sets with the Baire property.

It is very easy to construct a Lebesgue measurable function $f\colon \mathbb{R}^2 \to \mathbb{R}$, which is not Lebesgue supmeasurable and whose all sections are continuous except of at most one point. (Such functions were constructed by, e.g., J. W. Šragin [18, Remark 3] and M. Balcerzak [1, Corollary 1.4].) It was observed by Z. Grande and J. Lipiński [11], A. B. Kharazishvili [12], [13], and M. Balcerzak [1] that if we assume the Continuum Hypothesis or a similar set-theoretical assumption, then there are Lebesgue supmeasurable functions which are not Lebesgue measurable. An analogous construction can be made for the σ-field of sets with the Baire property [8].

One can ask whether some set-theoretical assumption is really necessary in the above theorems. It turns out that it is, both in the case of Lebesgue supmeasurability [17], and in the case of Baire supmeasurability [2], [7].

In this article, we examine the properties which guarantee the strong supmeasurability of functions of two variables. First, recall a very old theorem due to C. Carathéodory [4].

Theorem 3. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}$. If all vertical sections of f are continuous, and all horizontal sections of f are Lebesgue measurable, then f is Lebesgue supmeasurable.*

In 1978 Z. Grande proved (assuming Continuum Hypothesis) that there is a non-supmeasurable function $f\colon \mathbb{R}^2 \to [0,1]$ whose all vertical sections are approximately continuous (so they are derivatives) and whose all horizontal sections are measurable.

The function constructed by Z. Grande is nonmeasurable, too. So, we can ask, whether each Lebesgue measurable function $f\colon \mathbb{R}^2 \to \mathbb{R}$ whose all vertical sections are approximately continuous (or derivatives) is Lebesgue supmeasurable. Both these questions have been answered in the affirmative by Z. Grande [9], [10].

It should be noted that it is consistent with ZFC that each function $f\colon \mathbb{R}^2 \to \mathbb{R}$ whose all vertical sections are approximately continuous and whose all horizontal sections are Lebesgue measurable, is Lebesgue measurable [14]. Consequently, such functions are Lebesgue supmeasurable.

For the Baire property, we have a different case. E. Grande and Z. Grande showed in 1984 that there is a function $f\colon \mathbb{R}^2 \to [0,1]$, which is Lebesgue measurable and possesses the Baire property, whose all vertical sections are approximately continuous (so they are derivatives), whose all horizontal sections possess the Baire property, and which *is not* Baire supmeasurable [8].

The goal of this paper is to generalize the theorem of Z. Grande.

We start with a few auxiliary results. Despite of its simplicity, Lemma 4 is useful in applications.

Lemma 4. *Assume that all vertical sections of a function $f\colon X \times Y \to Z$ are derivatives. For each $n \in \mathbb{N}$ define $f_n\colon X \times Y \to Z$ by*

$$f_n(x,y) = \frac{\int_{J_n(y)} f_x(v)\,\nu(dv)}{\nu\big(J_n(y)\big)}\,. \tag{2}$$

Then the sequence (f_n) is pointwise convergent to f.

Proof. For each $(x,y) \in X \times Y$, since f_x is a derivative, we obtain

$$f(x,y) = f_z(y) = \lim_{n\to\infty} \frac{\int_{J_n(y)} f_x(v)\,\nu(dv)}{\nu\big(J_n(y)\big)} = \lim_{n\to\infty} f_n(x,y)\,.$$

□

Proposition 5. *Assume that all vertical sections of a function $f\colon X\times Y \to Z$ are derivatives. The following are equivalent:*

i) *f is strongly measurable,*

ii) *for each $J \in \mathscr{J}$, the mapping $h_J\colon X \to Z$ defined by*

$$h_J(x) = \int_J f_x(v)\,\nu(dv) \tag{3}$$

is strongly measurable,

iii) *for each $n \in \mathbb{N}$ the function $f_n\colon X \times Y \to Z$, defined by (2), is strongly measurable.*

Proof. The implication iii) $\Rightarrow$ i) is the consequence of Lemma 4 and the fact that the pointwise limit of a sequence of strongly measurable functions is strongly measurable (cf. [5, Proposition 5]), and the implication i)$\Rightarrow$ii) follows by the Fubini theorem. (Observe that since $\mathscr{J}$ is a net structure in Y and each net is countable, the measure ν is σ-finite.)

ii) $\Rightarrow$ iii) Assume that each mapping h_J is strongly measurable. Fix $n \in \mathbb{N}$. By Theorem 2, for each $J \in \mathscr{J}_n$ there exists an $A_J \in \mathcal{I}$ with the property that $h_J(X \setminus A_J)$ is a separable subspace of Z. Define $A^* = \left(\bigcup_{J \in \mathscr{J}_n} A_J \right) \times Y$. Then $A^* \in \widetilde{\mathcal{I}}$, and since

$$f_n\big((X \times Y) \setminus A^*\big) \subset \bigcup_{J \in \mathscr{J}_n} f_n\big((X \setminus A_J) \times J\big) = \bigcup_{J \in \mathscr{J}_n} \frac{h_J}{\nu(J)}(X \setminus A_J),$$

the set $f_n\big((X \times Y) \setminus A^*\big)$ is a separable subspace of Z. Moreover, for each open set $G \subset Z$ the inverse image

$$f_n^{-1}(G) = \bigcup_{J \in \mathscr{J}_n} \left\{ (x, y) \in X \times J \colon \frac{h_J(x)}{\nu(J)} \in G \right\} = \bigcup_{J \in \mathscr{J}_n} \left(\left(\frac{h_J}{\nu(J)} \right)^{-1} (G) \times J \right)$$

is measurable. So, the function f_n is measurable. Using again Theorem 2, we conclude that f_n is strongly measurable. $\square$

Theorem 6. *Assume that a function $f \colon X \times Y \to Z$ is strongly measurable, and all vertical sections of f are derivatives. Then f is strongly supmeasurable.*

Proof. For each $n \in \mathbb{N}$ define the function $f_n \colon X \times Y \to Z$ by (2). By Lemma 4, it suffices to show that each function f_n is strongly supmeasurable. (See Remark 2.)

Fix $n \in \mathbb{N}$ and let $\varphi \colon X \to Y$ be an $(\mathscr{M}, \mathscr{J})$-measurable function. Define the function $g \colon X \to Z$ by $g(x) = f_n\big(x, \varphi(x)\big)$. To show that g is strongly measurable, we will use Theorem 2.

By Proposition 5, f_n is strongly measurable. By Theorem 2, there exists an $A \in \mathcal{I}$ such that the set $f_n\big((X \setminus A) \times Y\big)$ is a separable subspace of Z. Since $g(X \setminus A) \subset f_n\big((X \setminus A) \times Y\big)$, we conclude that $g(X \setminus A)$ is a separable subspace of Z, too.

For each $J \in \mathscr{J}_n$, define the mapping h_J by (3). Observe that for each open set $G \subset Z$, we have

$$\begin{aligned} g^{-1}(G) &= \bigcup_{J \in \mathscr{J}_n} \Big(\varphi^{-1}(J) \cap \big\{ x \in X \colon f_n\big(x, \varphi(x)\big) \in G \big\} \Big) \\ &= \bigcup_{J \in \mathscr{J}_n} \left(\varphi^{-1}(J) \cap \left\{ x \in X \colon \frac{h_J(x)}{\nu(J)} \in G \right\} \right) \\ &= \bigcup_{J \in \mathscr{J}_n} \left(\varphi^{-1}(J) \cap \left(\frac{h_J}{\nu(J)} \right)^{-1} (G) \right). \end{aligned}$$

By Proposition 5, each mapping h_J is strongly measurable. Since the function φ is $(\mathscr{M}, \mathscr{J})$-measurable, we obtain $g^{-1}(G) \in \mathscr{M}$. Consequently, g is measurable, and by Theorem 2, the function g is strongly measurable. This completes the proof. $\square$

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