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# ON SET OF CONVERGENCE POINTS OF TRANSFINITE SEQUENCE OF QUASI-CONTINUOUS FUNCTIONS

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ABSTRACT. The aim of the paper is to characterize the class of sets of points at which a transfinite sequence of quasi-continuous real functions converges.

The notion of the convergence of $\omega_1$-sequences of functions was introduced by W. Sierpiński in [S1]. In his paper Sierpiński proved that the class of all continuous functions from $\mathbb{R}$ into $\mathbb{R}$ as well as the first class of Baire are closed under taking $\omega_1$-limit. This problem was also considered by many mathematicians for other classes of functions, see, e.g., P. Kostyrko [Ks], T. Šalát [Sa], A. Neubrunnová [N2] (who showed that a limit of a transfinite sequence of quasi-continuous functions is again quasi-continuous), J. Lipiński [Li], Z. Grande [Gr], P. Komjáth [Ko] and M. Dindoš [Di], who showed that the problem whether the classes of all approximately continuous functions and of all derivatives are closed with respect to transfinite limits cannot be solved in ZFC.

Our investigation takes a different direction, namely we are interested in characterizing the class of sets of points at which a transfinite sequence of real functions converges. The first result describing sets of convergence points is due to Hahn [Ha] and Sierpiński [S2]. It concerns pointwise convergence of sequences of continuous functions and it states that a subset $A$ of a Polish space $X$ is of type $\mathcal{F}_{\sigma\delta}$ if and only if there exists a sequence $\{f_n : n \in \mathbb{N}\} \subset \mathbb{R}^X$ of continuous functions convergent exactly at each point of $A$.

The same problem arises when we replace pointwise convergence with some other types of convergence. In [NW] the transfinite sequences of functions are examined. In particular, a characterization of a set of convergence points for

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a transfinite sequence of quasi-continuous functions defined on a real line is given. Here, we give a characterization of this set for quasi-continuous functions defined on a separable metric space. Moreover, we show that the assumption on a separability of a domain cannot be omitted.

We will use the standard terminology and notation. In particular, ordinal numbers will be identified with the set of their predecessors and cardinal numbers with the initial ordinals.Thus the first infinite cardinal $\omega$ is identified with the set of natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$. The cardinality of a set $A$ is denoted by $|A|$. The letter $\mathbb{R}$ stands for the set of real numbers. The cardinality of $\mathbb{R}$ is denoted by $\mathfrak{c}$.

Let $X$ be a metric space. For a subset $A$ of $X$ we denote by $\mathrm{int}(A)$, $\mathrm{cl}(A)$ and $\mathrm{fr}(A)$ the interior, closure and boundary of $A$, respectively. For $x \in X$ and $\varepsilon > 0$ let $B(x, \varepsilon)$ denote an open ball centred at $x$ with the radius $\varepsilon$. Throughout the paper the following abbreviations for some classes of subsets of $X$ are used:

$\mathcal{SO}(X)$ – semi-open subsets of $X$: $A \in \mathcal{SO}(X)$ iff $A \subset \mathrm{cl}(\mathrm{int}\, A)$ [Le];

$\mathcal{B}\mathrm{aire}(X)$ – sets with the Baire property;

$\mathcal{M}(X)$ – $\sigma$-ideal of meager (first-category) subsets of $X$.

For two arbitrary functions $f, g\colon X \to \mathbb{R}$ let $[f = g] = \{x \in X\colon f(x) = g(x)\}$. For a function $f\colon X \to \mathbb{R}$, by $C(f)$ we denote the set of all continuity points of $f$.

A transfinite sequence $\{x_\alpha\colon \alpha < \omega_1\}$ of real numbers is said to be convergent to $x \in \mathbb{R}$ iff for every positive number $\varepsilon$ there exists an ordinal $\alpha_0 < \omega_1$ such that $|x_\alpha - x| < \varepsilon$ for any $\alpha \geq \alpha_0$. Then we say that $x$ is a limit of a sequence $\{x_\alpha\colon \alpha < \omega_1\}$ and we denote this by $x = \lim_\alpha x_\alpha$.

**Remark 1.** A transfinite sequence $\{x_\alpha\colon \alpha < \omega_1\}$ is convergent if and only if there exists an ordinal $\alpha_0 < \omega_1$ such that $x_\alpha = x_{\alpha_0}$ for each $\alpha \geq \alpha_0$.

Let $\{f_\alpha\colon \alpha < \omega_1\} \subset \mathbb{R}^X$ be a transfinite sequence of real functions. Then $x \in X$ is said to be a convergence point of $\{f_\alpha\colon \alpha < \omega_1\}$ iff a corresponding sequence of values $\{f_\alpha(x)\colon \alpha < \omega_1\}$ is convergent. A function $f\colon X \to \mathbb{R}$ is said to be $\omega_1$-limit of $\{f_\alpha\colon \alpha < \omega_1\}$ iff $f(x) = \lim_\alpha f(x)_\alpha$ for every $x \in X$. Then we write $f = \lim_\alpha f_\alpha$.

We denote by $L^t(\{f_\alpha\colon \alpha < \omega_1\})$ a set of all convergence points of $\{f_\alpha\colon \alpha < \omega_1\}$, i.e.,

$$L^t(\{f_\alpha\colon \alpha < \omega_1\}) = \{x \in X\colon \lim_\alpha f_\alpha(x) \text{ exists}\}.$$

For a family of functions $\mathcal{F}(X) \subset \mathbb{R}^X$ define

$$\mathcal{L}^t(\mathcal{F}(X)) = \{L^t(\{f_\alpha\colon \alpha < \omega_1\})\colon \{f_\alpha\colon \alpha < \omega_1\} \subset \mathcal{F}(X)\}.$$

We examine the class $\mathcal{QC}(X)$ of *quasi-continuous* real functions defined on $X$. A function $f \in \mathcal{QC}(X)$ iff for every $p \in X$ and for every open sets $U \subset X$, $W \subset \mathbb{R}$ such that $p \in U$ and $f(p) \in W$ there exists a nonempty open set $G \subset U$ such that $f(G) \subset W$ (or, equivalently $f^{-1}(V) \in \mathcal{SO}(X)$ for every open $V \subset \mathbb{R}$, see, e.g., [N1, Theorem 1.1]). Denote also by $\mathcal{C}\mathrm{liq}(X)$ the collection of all *cliquish* functions, i.e., $f \in \mathcal{C}\mathrm{liq}(X)$ iff for every $x \in X$, its open neighbourhood $U$ and $\varepsilon > 0$ there exists a nonempty open set $G \subset U$ such that $\left|f(t) - f(t')\right| < \varepsilon$ for every $t, t' \in G$, [Th].

The following lemma is a key to a construction of quasi-continuous functions.

**Lemma 1.** (cf. Borsík [B1, Lemma 1]) *Let $X$ be an arbitrary metric space. For every nonempty nowhere dense closed set $F \subset X$ satisfying $F \subset \mathrm{cl}(G)$ for some nonempty open set $G \subset X$ there exists a collection $\{K_{n,m} \colon n \in \mathbb{N}, m \le n\}$ of nonempty open sets such that*

- (i) $\mathrm{cl}(K_{n,m}) \subset G \setminus F$ *for all* $n \in \mathbb{N}$ *and* $m \le n$;
- (ii) $\mathrm{cl}(K_{n,m}) \cap \mathrm{cl}(K_{i,j}) = \emptyset$ *for* $(n,m) \ne (i,j)$;
- (iii) *for every $x \notin F$ there exists an open neighbourhood $V$ of $x$ such that the set $\{(n,m) \colon \mathrm{cl}(K_{n,m}) \cap V \ne \emptyset\}$ has at most one element;*
- (iv) *for every $x \in F$, its open neighbourhood $V$ and a number $m \in \mathbb{N}$ there exists $n \ge m$ such that $K_{n,m} \cap V \ne \emptyset$.*

*Consequently, $F \subset \mathrm{cl}\left(\bigcup_{n \ge m} K_{n,m}\right)$ for each $m \in \mathbb{N}$ and both $F \cup \bigcup_{n \ge m} \mathrm{cl}(K_{n,m})$ and $F \cup \bigcup_{n \in \mathbb{N}, m \le n} \mathrm{cl}(K_{n,m})$ are closed in $G$.*

**Corollary 1.** *Let $X$ be an arbitrary metric space. For every nonempty nowhere dense closed set $F \subset X$ satisfying $F \subset \mathrm{cl}(G)$ for some nonempty open set $G \subset X$ there exist disjoint semi-open sets $S_0, S_1 \subset G \setminus F$ such that $G \setminus F = S_0 \cup S_1$ and $F \subset \mathrm{cl}(S_0) \cap \mathrm{cl}(S_1)$.*

Proof. By Lemma 1, there is a collection $\{K_{n,m} \colon n \in \mathbb{N}, m \le n\}$ of sets satisfying (i)–(iv). It is enough to take $S_0 = \bigcup_{n \in \mathbb{N}} \mathrm{cl}(K_{n,0})$ and $S_1 = G \Big\backslash \left(F \cup \bigcup_{n \in \mathbb{N}} \mathrm{cl}(K_{n,0})\right)$. ☐

**Lemma 2.** (cf. Borsík [B2]) *Let $X$ be an arbitrary metric space. For every cliquish function $g \colon X \to \mathbb{R}$ there are quasi-continuous functions $s, t \colon X \to \mathbb{R}$ such that $g = s + t$.*

**Theorem 1.** *Let $X$ be a separable metric space. The class $\mathcal{L}^t\big(\mathcal{QC}(X)\big)$ is equal to the family of all sets of the form $(G \setminus M) \cup N$, where $G$ is open, $M \in \mathcal{M}(X)$ and $N$ is nowhere dense. Therefore, $\mathcal{L}^t\big(\mathcal{QC}(X)\big) \subset \mathcal{B}\mathrm{aire}(X)$.*

P r o o f . First, let $A = L^t(\{f_\alpha\colon \alpha < \omega_1\})$ for some sequence $\{f_\alpha\colon \alpha < \omega_1\} \subset \mathcal{QC}(X)$ and $f = \lim_\alpha f_\alpha|_A$. Consider a set $G = \operatorname{int}(\operatorname{cl}(A))$. Then $A = (G \cap A) \cup (A \setminus G)$ and $A \setminus G$ is nowhere dense. Choose a countable set $C_0 \subset G \cap A$ dense in $G \cap A$. Then there is $\alpha_0 < \omega_1$ such that

$$f_\alpha|_{C_0} = f|_{C_0} \text{ for every } \alpha \geq \alpha_0 . \tag{$*$}$$

Notice that $C(f_{\alpha_0})$ is residual, by quasi-continuity of $f_{\alpha_0}$ (see, e.g., [Le]). Now, fix $x \in G \cap C(f_{\alpha_0})$ and $\alpha \geq \alpha_0$. We will show that $f_\alpha(x) = f_{\alpha_0}(x)$. By quasi-continuity of $f_\alpha$ at $x$, for every $k < \omega$ there is a non-empty open set $V_k \subset G \cap B\left(x, \frac{1}{k+1}\right)$ such that $f_\alpha(V_k) \subset \left(f_\alpha(x) - \frac{1}{k+1}, f_\alpha(x) + \frac{1}{k+1}\right)$. Since $C_0$ is dense in $G$ there is $x_k \in C_0 \cap V_k$ such that $f_\alpha(x_k) \in \left(f_\alpha(x) - \frac{1}{k+1}, f_\alpha(x) + \frac{1}{k+1}\right)$. Then we get a sequence $\{x_k\colon k < \omega\} \subset C_0$ such that $\lim_k x_k = x$ and $\lim_k f_\alpha(x_k) = f_\alpha(x)$. Consequently, by $(*)$ and continuity of $f_{\alpha_0}$ at $x$ we have

$$f_\alpha(x) = \lim_k f_\alpha(x_k) = \lim_k f_{\alpha_0}(x_k) = f_{\alpha_0}(x)$$

for every $\alpha \geq \alpha_0$, so $x \in A$.

Finally, since $G \cap C(f_{\alpha_0}) \subset G \cap A$ and $C(f_{\alpha_0})$ is residual in $G$, we have $G \setminus A \in \mathcal{M}(X)$. Putting $G = \operatorname{int}(\operatorname{cl}(A))$, $M = G \setminus A$, $N = A \setminus \operatorname{int}(\operatorname{cl}(A))$ we get the sets we claimed.

To prove "$\supset$", fix $A = (G \setminus M) \cup N$, where $G$ is open, $M \in \mathcal{M}(X)$, $M \subset G$ and $N$ is nowhere dense. We may assume that $N \cap G = \emptyset$ and $G$ is a regular open set (i.e., $G = \operatorname{int}(\operatorname{cl}(G))$ ), since $\operatorname{int}(\operatorname{cl}(G)) \setminus G$ is nowhere dense and may be consider as a subset of $N$. Let $M = \bigcup_{i \in \mathbb{N}} F_i$ for an increasing sequence $\{F_i\colon i \in \mathbb{N}\}$ of nowhere dense subsets of $G$. We need to find $\{f_\alpha\colon \alpha < \omega_1\} \subset \mathcal{QC}(X)$ such that $A = L^t(\{f_\alpha\colon \alpha < \omega_1\})$.

First, consider a function $g\colon G \to \mathbb{R}$ given as follows:

$$g(x) = \begin{cases} 1 & \text{for } x \in F_0 , \\ \frac{1}{i} & \text{for } x \in F_i \setminus F_{i-1} \text{ and } i \geq 1 , \\ 0 & \text{otherwise} . \end{cases}$$

It is easy to see that $g \in \mathcal{C}\mathrm{liq}(G)$. Moreover, $[g = 0] = G \setminus M$. By Lemma 2, there are quasi-continuous functions $g_0, g_1\colon G \to \mathbb{R}$ such that $g = g_0 - g_1$. Take a bijection $\varphi\colon \omega_1 \to 2 \times \omega_1$, $\varphi = (\varphi_0, \varphi_1)$ and for each $\alpha < \omega_1$ put

$$f_\alpha|_G = \varphi_0(\alpha) \cdot g_0 + \left(1 - \varphi_0(\alpha)\right) \cdot g_1 .$$

Then $\{f_\alpha|_G\colon \alpha < \omega_1\} \subset \mathcal{QC}(G)$ and $L^t(\{f_\alpha|_G\colon \alpha < \omega_1\}) = G \setminus M$.

If $X = G$, the proof is complete. Suppose $X \setminus G \neq \emptyset$. Since $G$ is a regular open set, $X \setminus G \in \mathcal{SO}(X)$. It follows that $X \setminus G = \operatorname{cl}\left(\operatorname{int}(X \setminus G)\right) = \operatorname{int}(X \setminus G) \cup \operatorname{fr}(G)$

and $\mathrm{int}(X \setminus G) \neq \emptyset$. Let $E = \mathrm{cl}(N) \cup \mathrm{fr}(G)$. Then $E$ is nowhere dense and $E \subset \mathrm{cl}\big(\mathrm{int}(X \setminus G)\big)$. If $N$ is nonempty, by Corollary 1, there are semi-open sets $S_0, S_1$ such that:

(i) $S_0 \cap S_1 = \emptyset$,

(ii) $\mathrm{int}(X \setminus G) \setminus E = S_0 \cup S_1$,

(iii) $E \subset \mathrm{cl}(S_0) \cap \mathrm{cl}(S_1)$.

Define $h, k \in \mathcal{QC}(X \setminus G)$ as follows:

$$h(x) = \begin{cases} 0 & \text{if } x \in S_0 \cup (E \cap A), \\ 1 & \text{if } x \in S_1 \cup (E \setminus A), \end{cases}$$

and

$$k(x) = \begin{cases} 1 & \text{if } x \in S_0, \\ 0 & \text{if } x \in S_1 \cup E. \end{cases}$$

If $N$ is empty, take $h(x) = 0$ and $k(x) = 1$. Then for $f_\alpha|_{X \setminus G}$ defined by

$$f_\alpha|_{X \setminus G} = \varphi_0(\alpha) \cdot h + \big(1 - \varphi_0(\alpha)\big) \cdot k$$

we have $\{f_\alpha|_{X \setminus G} \colon \alpha < \omega_1\} \subset \mathcal{QC}(X \setminus G)$ and $L^t\big(\{f_\alpha|_{X \setminus G} \colon \alpha < \omega_1\}\big) = N$. Since $G$ is a regular open set, $f_\alpha = f_\alpha|_G \cup f_\alpha|_{X \setminus G}$ are quasi-continuous. So, a sequence $\{f_\alpha \colon \alpha < \omega_1\}$ has all desired properties. $\square$

It turns out that under the Continuum Hypothesis in Theorem 1 the assumption on $X$ to be separable cannot be dropped. To see this, consider the following example.

Example 1. Let $X$ be a space of all $\mathcal{B}_1$ real functions defined on $\mathbb{R}$ equipped with the uniform convergence metric $\varrho$, i.e., for $f, g \in X$

$$\varrho(f, g) = \sup_{x \in \mathbb{R}} \hat{\varrho}\big(f(x), g(x)\big),$$

where $\hat{\varrho}(x, y) = \min\{1, |x - y|\}$ for $x, y \in \mathbb{R}$. We will find a subset $M$ of $X$ such that $M \in \mathcal{L}^t\big(\mathcal{QC}(X)\big)$ and $M$ cannot be written in the form we claimed in Theorem 1.

For an arbitrary $f \in X$ and a number $\varepsilon > 0$ consider a family $A = \{f + (\varepsilon/2) \cdot \chi_{\{x\}} \colon x \in \mathbb{R}\} \subset B(f, \varepsilon)$, where $\chi_{\{x\}}$ is a characteristic function of $\{x\} \subset \mathbb{R}$. Then $A$ is a discrete subspace of $X$ and $|A| = \mathfrak{c}$. It follows that there exists a family of pairwise disjoint open subsets of $B(f, \varepsilon)$ which has a cardinality $\mathfrak{c}$. Consequently, every subset of $X$ of a power less then $\mathfrak{c}$ is nowhere dense in $X$.

Now, fix a positive number $n \in \mathbb{N}$ and consider

$$\mathcal{P}_n = \Big\{A \subset X \colon \varrho(f, g) \geq \frac{1}{n} \text{ for any } f, g \in A\Big\}.$$

Denote by $P_n$ a maximal element of $\mathcal{P}_n$. Define $M = \bigcup_{n>0} P_n$. Then $M \in \mathcal{M}(X)$ and it is dense in $X$. Since $X$ is complete, $M$ cannot be written as in Theorem 1 but $M \in \mathcal{L}^t\big(\mathcal{QC}(X)\big)$. To see this, first observe that $|M| = \mathfrak{c}$ by the previous remarks. Now, assume the Continuum Hypothesis. Then $M = \{x_\alpha : \alpha < \omega_1\}$. For any $\alpha < \omega_1$ define $M_\alpha = \{x_\beta : \beta \le \alpha\}$ which is nowhere dense. By Corollary 1 we have semi-open sets $S^0_\alpha$, $S^1_\alpha$ such that

(i) $S^0_\alpha \cap S^1_\alpha = \emptyset$,

(ii) $X \setminus \mathrm{cl}(M_\alpha) = S^0_\alpha \cup S^1_\alpha$,

(iii) $M_\alpha \subset \mathrm{cl}(S^0_\alpha) \cap \mathrm{cl}(S^1_\alpha)$.

Define $g_\alpha, h_\alpha \in \mathcal{QC}(X)$ as follows:

$$g_\alpha(x) = \begin{cases} 0 & \text{if } x \in S^0_\alpha \cup M_\alpha\,, \\ 1 & \text{if } x \in S^1_\alpha \cup \big(\mathrm{cl}(M_\alpha) \setminus M_\alpha\big)\,, \end{cases}$$

and

$$h_\alpha(x) = \begin{cases} 1 & \text{if } x \in S^0_\alpha\,, \\ 0 & \text{if } x \in S^1_\alpha \cup \mathrm{cl}(M_\alpha)\,. \end{cases}$$

Take an arbitrary bijection $\varphi : \omega_1 \to 2 \times \omega_1$, $\varphi = (\varphi_0, \varphi_1)$ and define

$$f_\alpha = \varphi_0(\alpha) \cdot g_{\varphi_1(\alpha)} + \big(1 - \varphi_0(\alpha)\big) \cdot h_{\varphi_1(\alpha)}\,.$$

Then $\{f_\alpha : \alpha < \omega_1\} \subset \mathcal{QC}(X)$ and $L^t\big(\{f_\alpha : \alpha < \omega_1\}\big) = M$.

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