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PROXIMITIES DEFINED BY OPEN COVERS AND CONVERGENCE OF NETS OF MULTIVALUED MAPS

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ABSTRACT. Proximities defined by open covers and convergence of nets of multivalued maps are investigated.

In this paper we will consider the following conditions on the relation δ defined on the power set of Y:

(1) $A\ \delta\ B \Longrightarrow B\ \delta\ A$;

(2) $(A \cup B)\ \delta\ C \Longleftrightarrow A\ \delta\ C \vee B\ \delta\ C$;

(3) $A\ \delta\ B \Longrightarrow A \neq \emptyset \wedge B \neq \emptyset$;

(4) $A \cap B \neq \emptyset \Longrightarrow A\ \delta\ B$;

(5) $A \not\delta\ B$ implies that there exists an $E \subset Y$ such that $A \not\delta\ E$ and $(Y \setminus E) \not\delta\ B$;

(6) $p \not\delta\ B$ implies that there exists an $E \subset Y$ such that $p \not\delta\ E$ and $(Y \setminus E) \not\delta\ B$;

(7) $(A\ \delta\ B \wedge \forall\, b \in B \colon b\ \delta\ C) \Longrightarrow A\ \delta\ C$.

Definition 1. ([2], [7]). A relation δ on the power set of Y having the properties (1)–(4), is called: a *Efremovic* (EF-), a *regular* (R-) and *Lodato* (LO-) *proximity* on Y if it satisfies (5), (6), and (7), respectively.

It is known that EF implies R and LO but LO and R are independent [2].

Let $(Y, \mathfrak{U})$ be a uniform space. The relation $\delta_{\mathfrak{U}}$ defined by

$$A\ \delta_{\mathfrak{U}}\ B \Longleftrightarrow \forall_{U \in \mathfrak{U}} (A \times B) \cap U \neq \emptyset$$

is an EF-proximity compatible with the topology in Y ([4], [7]).

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In the sequel, various topological spaces will be considered on which none separation axioms will be assumed if it is not explicitly stated. Thus a regular (normal, compact, paracompact) space need not be Hausdorff.

Let Y be a topological space and let $\mathcal{A}$ be an open cover of Y.

Definition 2. The set $\mathrm{St}(A, \mathcal{A}) = \bigcup\{U \in \mathcal{A} \colon U \cap A \neq \emptyset\}$ is called a *star* of a set A with respect to a cover $\mathcal{A}$ of a space Y.

Lemma 1. [3] *If Y is a regular space and A is a subset of Y, then*

$$\overline{A} = \bigcap_{\mathcal{A}} \mathrm{St}(A, \mathcal{A}),$$

where the intersection runs through all open covers $\mathcal{A}$ of Y.

Definition 3. In the topological space Y we define the relation $\delta^\star$ by

$$A\ \delta^\star\ B \iff A \cap \mathrm{St}(B, \mathcal{A}) \neq \emptyset \quad \text{for every open cover } \mathcal{A} \text{ of } Y.$$

Theorem 1. *Let Y be a regular space. The relation $\delta^\star$ is an LO and R-proximity compatible with the topology in Y.*

Proof. Let $A, B \subset Y$ be such that $A\ \delta^\star\ B$ and let $\mathcal{A}$ be any open cover of the space Y. From the assumption $A \cap \mathrm{St}(B, \mathcal{A}) \neq \emptyset$. Then exists a set $U \in \mathcal{A}$ such that $A \cap U \neq \emptyset$ and $B \cap U \neq \emptyset$. So $B \cap \mathrm{St}(A, \mathcal{A}) \neq \emptyset$ for every open cover $\mathcal{A}$ of space Y. That means $B\ \delta^\star\ A$.

Now, we show that the following condition

$$(A \cup B)\ \delta^\star\ C \iff A\ \delta^\star\ C \vee B\ \delta^\star\ C$$

is satisfied. Let $A, B, C \subset Y$ and let $(A \cup B)\ \delta^\star\ C$. Suppose that $A\ \not\delta^\star\ C$ and $B\ \not\delta^\star\ C$. Then there exist the open covers $\mathcal{A}$ and $\mathcal{B}$ of the space Y such that $A \cap \mathrm{St}(C, \mathcal{A}) = \emptyset$ and $B \cap \mathrm{St}(C, \mathcal{B}) = \emptyset$. We consider the family of sets

$$\mathcal{G} = \{U \cap V \colon U \in \mathcal{A},\ V \in \mathcal{B}\}.$$

Then $\mathcal{G}$ is an open cover of Y which refines $\mathcal{A}$ and $\mathcal{B}$. So $A \cap \mathrm{St}(C, \mathcal{G}) = \emptyset$ and $B \cap \mathrm{St}(C, \mathcal{G}) = \emptyset$. That means $(A \cup B) \cap \mathrm{St}(C, \mathcal{G}) = \emptyset$. Hence $(A \cup B)\ \not\delta^\star\ C$. Now, we assume that $A\ \delta^\star\ C$ or $B\ \delta^\star\ C$. Let $\mathcal{A}$ be any open cover of the space Y. Assume that $A\ \delta^\star\ C$. Then $A \cap \mathrm{St}(C, \mathcal{A}) \neq \emptyset$, so $(A \cup B) \cap \mathrm{St}(C, \mathcal{A}) \neq \emptyset$. That means $(A \cup B)\ \delta^\star\ C$. Analogously for $B\ \delta^\star\ C$.

Let $A, B \subset Y$ be such that $A\ \delta^\star\ B$. Then $A \cap \mathrm{St}(B, \mathcal{A}) \neq \emptyset$ for every open cover $\mathcal{A}$ of the space Y, hence $A \neq \emptyset$ and $\mathrm{St}(B, \mathcal{A}) \neq \emptyset$. Thus $B \neq \emptyset$.

Now, we will show that (4) holds.

Suppose $A \cap B \neq \emptyset$. Since $B \subset \mathrm{St}(B, \mathcal{A})$ for every open cover $\mathcal{A}$ of Y then we obtain $A \cap \mathrm{St}(B, \mathcal{A}) \neq \emptyset$, so $A\ \delta^\star\ B$.

Now, we will show that

$$(A\ \delta^\star\ B \wedge \forall\, b \in B \colon b\ \delta^\star\ C) \Longrightarrow A\ \delta^\star\ C\,.$$

Let $A, B, C \subset Y$, $A\ \delta^\star\ B$ and let $b\ \delta^\star\ C$ for every $b \in B$. If $b\ \delta^\star\ C$, then for every open cover $\mathcal{A}$ of Y we have $\{b\} \cap \mathrm{St}(C, \mathcal{A}) \neq \emptyset$ for each $b \in B$. So $b \in \mathrm{St}(C, \mathcal{A})$ for every open cover $\mathcal{A}$ of Y. Consequently, $b \in \bigcap_{\mathcal{A}} \mathrm{St}(C, \mathcal{A})$ for $b \in B$. Since Y is a regular space, then in accordance with Lemma 1 we have $\bigcap_{\mathcal{A}} \mathrm{St}(C, \mathcal{A}) = \overline{C}$. That is $b \in \overline{C}$ for each $b \in B$. That means $B \subset \overline{C}$. Since $A\ \delta^\star\ B$ and $B \subset \overline{C}$, so $A\ \delta^\star\ \overline{C}$. Hence, since $\mathrm{St}(C, \mathcal{A}) = \mathrm{St}(\overline{C}, \mathcal{A})$, it is clear that $A\ \delta^\star\ C$.

We have shown that $\delta^\star$ is LO-proximity.

Now, we will show that the following condition

$$p\ \not\delta^\star\ B \implies \exists_{E \subset Y} \big[p\ \not\delta^\star\ E \wedge (Y \setminus E)\ \not\delta^\star\ B\big]$$

is satisfied. If $p\ \not\delta^\star\ B$, then there exists an open cover $\mathcal{G}$ of Y such that $p \notin \mathrm{St}(B, \mathcal{G})$. Hence $p \notin \overline{B}$. There exist open sets $U_1, U_2, V_1 \subset Y$ such that $p \in U_2 \subset \overline{U}_2 \subset U_1$, $\overline{B} \subset V_1$ and $U_1 \cap V_1 = \emptyset$. The family

$$\mathcal{G}' = \big\{U_1, V_1, Y \setminus [\overline{U}_2 \cup \overline{B}]\big\}$$

forms an open cover of Y. Of course, $p \notin V_1 = \mathrm{St}(B, \mathcal{G}')$.

Let $E = Y \setminus \overline{U}_1$. Then $p \notin Y \setminus \overline{U}_2 = \mathrm{St}(E, \mathcal{G}')$. So $p\ \not\delta^\star\ E$. Since $\overline{U}_1 \cap V_1 = \emptyset$ so $\overline{U}_1 \cap \mathrm{St}(B, \mathcal{G}') = \emptyset$. Hence $\overline{U}_1\ \not\delta^\star\ B$. From definition of the set E we have $(Y \setminus E)\ \not\delta^\star\ B$.

We have shown that $\delta^\star$ is R-proximity.

Now we will show, that the relation $\delta^\star$ is compatible with the topology in Y.

For a set $B \subset Y$ by $\overline{B}^{\delta^\star}$ we denote the closure determined by $\delta^\star$.

Assume that $p \in \overline{B} = \bigcap\limits_{\mathcal{A}} \mathrm{St}(B, \mathcal{A})$, where the intersection runs through all open covers $\mathcal{A}$ of Y. That means that $\{p\} \cap \mathrm{St}(B, \mathcal{A}) \neq \emptyset$ for each $\mathcal{A}$. From the definition of relation $\delta^\star$ we have $\delta^\star\ B$ which means $p \in \overline{B}^{\delta^\star}$. Thus $\overline{B} = \overline{B}^{\delta^\star}$. $\square$

The assumption of the regularity is essential.

Example 1. We take the real line $\mathbb{R}$ with the topology $\mathfrak{T}$ defined by

$$\mathfrak{T} = \big\{\emptyset\,,\ \mathbb{R}\,,\ \mathbb{R} \setminus \{a\}\,,\ \mathbb{R} \setminus \{c\}\,,\ \mathbb{R} \setminus \{a, c\}\big\}\,,$$

where $a, c \in \mathbb{R}$, $a < c$. Let us notice that there are not two open, non-empty and disjoint sets in this space. Hence $(\mathbb{R}, \mathfrak{T})$ is not a regular space.

Now, we will show that the relation $\delta^\star$ is not an LO-proximity.

Let $b \in \mathbb{R}$ and $a < b < c$. We take the open cover $\mathfrak{B} = \{\mathbb{R} \setminus \{a\}, \mathbb{R} \setminus \{c\}\}$ of space $\mathbb{R}$. Since $\mathrm{St}(c, \mathfrak{B}) = \mathbb{R} \setminus \{a\}$, so $a \notin \mathrm{St}(c, \mathfrak{B})$. Then $a \not\delta^\star c$. Let us notice that b belongs to each non-empty open set, so it belongs to $\mathrm{St}(a, \mathfrak{A})$ and $\mathrm{St}(c, \mathfrak{A})$, where $\mathfrak{A}$ is any open cover of $\mathbb{R}$. So $a \ \delta^\star \ b$ and $b \ \delta^\star \ c$. Thus the relation $\delta^\star$ is not an LO-proximity.

Let us notice, that $\delta^\star$ need not be the EF-proximity.

Example 2. Let Y be the Niemytzki plane and let T_E be the natural topology on the real line. The symbols $\mathrm{Int}_{T_E} A$ and $\overline{A}^{T_E}$ are used to denote the interior and the closure of the set A in $(\mathbb{R}, T_E)$. By L_1 we denote the line $y = 0$. For each $p \in L_1$ and $i \in \mathbb{N}$ let $K_{p,i}$ be the set of points belonging to $L_2 = Y \setminus L_1$ contained in the ball with radius $1/i$ tangent to L_1 at the point p. Then we put $U_i(p) = K_{p,i} \cup \{p\}$. Let

$$A = \{p = (x, 0) \colon x \in \mathbb{Q}\}, \ B = \{p = (y, 0) \colon y \in \mathbb{IQ}\}.$$

We will show that the condition (5) is not satisfied.

Let us notice that for the open cover

$$\mathcal{A} = \{L_2\} \cup \{U_i(p) \colon p \in L_1, \ i \in \mathbb{N}\}$$

of the space Y we have $A \cap \mathrm{St}(B, \mathcal{A}) = \emptyset$ so $A \not\delta^\star B$. Let $E \subset Y$. We will show that $(Y \setminus E) \ \delta^\star \ A$ or $B \ \delta^\star \ E$. We consider two situations.

At first, if $(Y \setminus E) \ \delta^\star \ A$, then we are done. Secondly, if $(Y \setminus E) \not\delta^\star A$, then there exists an open cover $\mathcal{G}$ of Y such that $(Y \setminus E) \cap \mathrm{St}(A, \mathcal{G}) = \emptyset$. Hence $\mathrm{St}(A, \mathcal{G}) \subset E$. It is known $A \subset \mathrm{St}(A, \mathcal{G})$. So for each $x \in \mathbb{Q}$ there exists a set $W \in \mathcal{G}$ containing a point $(x, 0)$ such that $A \cap W \neq \emptyset$. So there exists $i_x \in \mathbb{N}$ such that $U_{i_x}(x, 0) \subset W$. Of course, $\bigcup_{x \in \mathbb{Q}} U_{i_x}(x, 0) \subset \mathrm{St}(A, \mathcal{G}) \subset E$.

We choose any open cover $\mathcal{B}$ of Y. For every point $p = (y, 0) \in B$ there exists a set $V \in \mathcal{B}$ such that $p \in V$. So there exists $j_y \in \mathbb{N}$ such that $U_{j_y}(p) \subset V$. Hence $\bigcup_{y \in \mathbb{IQ}} U_{j_y}(p) \subset \mathrm{St}(B, \mathcal{B})$. We denote by

$$M_n = \{y \in \mathbb{IQ} \colon j_y = n\}.$$

Then $\mathbb{IQ} = \bigcup_n M_n$. Because the set $\mathbb{IQ}$ is of the second category, there exists $m \in \mathbb{N}$ such that M_m is nowhere dense. That means, that there exists a point $x_0 \in \mathbb{Q}$ with $x_0 \in \mathrm{Int}_{T_E} \overline{M_m}^{T_E}$. There exists a sequence $\{y_k \colon k \in \mathbb{N}\}$ such that $y_k \in M_k$ for $k \in \mathbb{N}$ and $|y_k - x_0| \to 0$, $k \to \infty$. That is $U_{j_{y_k}}((y_k, 0)) \subset \mathrm{St}(B, \mathcal{B})$. So for sufficiently large k we have $\left(\bigcup_{j_{y_k}} U_{j_{y_k}}(p_k)\right) \cap U_{i_{x_0}}(x_0) \neq \emptyset$, where $p_k = (y_k, 0)$. Consequently, we have $\mathrm{St}(B, \mathcal{B}) \cap E \neq \emptyset$ for each open cover $\mathcal{B}$ of Y. So $B \ \delta^\star \ E$.

This means that the relation $\delta^\star$ is not the EF-proximity.

In the consequence, the proximity $\delta^\star$ is not determined by a compatible uniformity in the Niemytzki plane.

Definition 4. Let $\mathfrak{U}$ be a uniformity in Y. For each set $V \in \mathfrak{U}$ the family $\{V[p] \colon p \in Y\}$ is called a *uniform cover*.

Lemma 2. *Let $(Y, \mathfrak{U})$ be a uniform space. Then for each set $U \in \mathfrak{U}$ there exists a uniform cover $\mathcal{G}$ of Y such that $\mathrm{St}(M, \mathcal{G}) \subset U[M]$ for each $M \subset Y$.*

The standard proof of this lemma is omitted.

Definition 5. [5] A uniform space $(Y, \mathfrak{U})$ is said to have the *Lebesgue property* if for each open cover $\mathcal{A}$ of Y there is a uniform cover which refines $\mathcal{A}$.

The connections between $\delta^\star$ and $\delta_{\mathfrak{U}}$ are given by the following

Theorem 2. *Let Y be a Tychonoff space and let $\mathfrak{U}$ be a uniformity compatible with the topology in Y. Then*

(a) $\delta_{\mathfrak{U}} \le \delta^\star$;

(b) *if $(Y, \mathfrak{U})$ has Lebesgue property, then $\delta^\star = \delta_{\mathfrak{U}}$.*

Proof. Let $A, B \subset Y$ and $A \ \delta^\star \ B$ and let $U \in \mathfrak{U}$. There exists an open uniform cover $\mathcal{A}$ of Y such that $\mathrm{St}(M, \mathcal{A}) \subset U[M]$ for each $M \subset Y$. If $A \ \delta^\star \ B$, then $A \cap \mathrm{St}(B, \mathcal{A}) \neq \emptyset$. Moreover, $\mathrm{St}(B, \mathcal{A}) \subset U[B]$, so $A \cap U[B] \neq \emptyset$ for each $U \in \mathfrak{U}$. Hence $A \ \delta_{\mathfrak{U}} \ B$.

Now, we will prove the condition (b). From the condition (a) we have $\delta_{\mathfrak{U}} \le \delta^\star$. Now, let $A \ \delta_{\mathfrak{U}} \ B$. Let $\mathcal{A}$ be any open cover of Y. Because the space $(Y, \mathfrak{U})$ has the Lebesgue property, some $U \in \mathfrak{U}$ can be chosen such that the family $\mathcal{B} = \{U[y] \colon y \in Y\}$ is a refinement of $\mathcal{A}$. Moreover $U[B] = \bigcup_{y \in B} U[y] \subset \mathrm{St}(B, \mathcal{B}) \subset \mathrm{St}(B, \mathcal{A})$. Since $A \cap U[B] \neq \emptyset$, so $A \cap \mathrm{St}(B, \mathcal{A}) \neq \emptyset$ for each open cover $\mathcal{A}$ of Y. That means $A \ \delta^\star \ B$. We have shown that $\delta^\star = \delta_{\mathfrak{U}}$. □

Theorem 3. [5] *Let Y be a uniformisable Hausdorff topological space, then Y possesses a uniformity compatible with the topology of Y for which Y has the Lebesgue property if and only if Y is paracompact. Under this condition, this uniformity is the finest of all uniformities compatible with the topology of Y, and is unique.*

Corollary. *If Y is a paracompact Hausdorff space, $\mathfrak{U}$—the finest uniformity compatible with the topology of Y, then $\delta^\star = \delta_{\mathfrak{U}}$.*

Example 2 shows that in the above corollary the assumption of the paracompactness is essential.

Let X be a non-empty set and let (Y, δ) be any proximity space of the type EF, LO or R.

Definition 6. A net $\{F_j : j \in J\}$ of multivalued maps $F_j : X \to Y$ is said to be *convergent in a sense of Leader* to a multivalued map $F : X \to Y$ (shortly $F_j \xrightarrow{LC} F$) if

$$\forall_{A\subset X, B\subset Y}\big[F(A) \not\delta B \Longrightarrow \exists_{j_0\in J}\forall_{j\geq j_0} F_j(A) \not\delta B\big]\,.$$

For the sequences of functions the above gives the convergence introduced by Leader [2].

The convergence in the sense of Leader essentially depends on the relation.

Example 3. Let $\mathbb{R}$ be the real line with the relation $\delta^\star$ and with the relation δ which is defined by

$$A\,\delta\,B \iff \big[\operatorname{dist}(A,B) = 0 \land A \neq \emptyset \neq B\big]\,,$$

where $\operatorname{dist}(A,B) = \inf\big\{|a-b| : a \in A,\ b \in B\big\}$.

We define functions $f, f_n : \mathbb{R} \to \mathbb{R}$ as follows: $f(x) = x^2$ and $f_n(x) = x^2 + \frac{1}{n}$ for every $x \in \mathbb{R}$ and $n \in \mathbb{N}$. The relations $\delta^\star$ and δ are compatible with the topology in $\mathbb{R}$, the sequence $\{f_n : n \in \mathbb{N}\}$ is convergent in the sense of Leader to the function $f : \mathbb{R} \to \mathbb{R}$ with respect to δ, but it is not convergent in the sense of Leader to f with respect to $\delta^\star$.

Definition 7. [3] Let X be a non-empty set, Y—a topological space. A net $\{F_j : j \in J\}$ of multivalued maps $F_j : X \to Y$ is said to be *upper strongly uniformly convergent to a multivalued map* $F : X \to Y$ if for each open cover $\mathcal{A}$ of Y there exists $j_0 \in J$ such that $F_j(x) \subset \operatorname{St}(F(x), \mathcal{A})$ for every $j \geq j_0$ and $x \in X$.

The above upper strong uniform convergence (Def. 7) is weaker than the strong convergence considered in [6].

In a topological space (Y, τ) we denote by 2^Y the family of all non-empty subsets of Y. For an open set $U \subset Y$ we write

$$U^+ = \{B \in 2^Y : B \subset U\}\,.$$

The family $\mathcal{B}^+ = \{U^+ : U \in \tau\}$ forms a base of the upper Vietoris topology. This topology will be denoted by τ^+. We will write $A \in \tau^+\text{-}\lim A_j$ if the net $\{A_j : j \in J\}$ converges to A in the space $\big(2^Y, \tau^+\big)$.

We have obtained the following connections between the Leader convergence and some other ones. Let X be a non-empty set and let (Y, δ) be any proximity space of the type EF or R.

THEOREM 4. *If a multivalued map $F\colon X \to Y$ has compact values, then the Leader convergence of a net $\{F_j\colon j \in J\}$ to F implies the τ^+-pointwise convergence.*

P r o o f. Let us take $x_0 \in X$ and $U^+ \in \mathcal{B}^+$ such that $F(x_0) \in U^+$. This means that U is an open set and $F(x_0) \subset U$. Of course $F(x_0) \cap (Y \setminus U) = \emptyset$. Furthermore, the values of F are compact, so $F(x_0) \not\delta (Y \setminus U)$. By the Leader convergence there exists $j_0 \in J$ such that $F_j(x_0) \not\delta (Y \setminus U)$ for each $j \geq j_0$. Hence $F_j(x_0) \cap (Y \setminus U) = \emptyset$. This means $F_j(x_0) \subset U$. Thus $F \in \tau^+\text{-}\lim F_j$. □

THEOREM 5. *Let X be a non-empty set and Y a T_4 space. If a net $\{F_j\colon j \in J\}$ of multivalued maps $F_j\colon X \to Y$ is upper strongly uniformly convergent to a multivalued map $F\colon X \to Y$, then it is convergent to F in the sense of Leader with respect to $\delta^\star$.*

P r o o f. Let $A \subset X$, $B \subset Y$ and let $F(A) \not\delta^\star B$. Then $\overline{F(A)} \not\delta^\star \overline{B}$. Since Y is a normal space, so there exist open sets $U_1, U_2 \subset Y$ such that $\overline{F(A)} \subset U_1$, $\overline{B} \subset U_2$ and $U_1 \cap U_2 = \emptyset$. The family $\mathcal{A} = \{U_1,\, U_2,\, Y \setminus \overline{B \cup F(A)}\}$ forms an open cover of Y. There exists j_0 such that for each $j \geq j_0$ and $x \in X$ we have $F_j(x) \subset \operatorname{St}\bigl(F(x), \mathcal{A}\bigr)$. Hence $F_j(A) \subset \operatorname{St}\bigl(F(A), \mathcal{A}\bigr)$ for each $j \geq j_0$. Since $\operatorname{St}\bigl(F(A), \mathcal{A}\bigr) = U_1$ and $\operatorname{St}(B, \mathcal{A}) = U_2$ and $U_1 \cap U_2 = \emptyset$, then $F_j(A) \cap \operatorname{St}(B, \mathcal{A}) = \emptyset$ for $j \geq j_0$. Hence $F_j(A) \not\delta^\star B$ for each $j \geq j_0$. This means, that the net $\{F_j\colon j \in J\}$ is convergent in the sense of Leader to F with respect to $\delta^\star$. □

LEMMA 3. *Let X be a topological space, Y—a regular one and let $x_0 \in X$. If a multivalued map $F\colon X \to Y$ has compact values, then the following conditions are equivalent:*

(a) *F is lower semi-continuous at $x_0 \in X$;*

(b) *for each open cover $\mathcal{A}$ of Y there exists a neighbourhood U of $x_0 \in X$ such that $F(x_0) \subset \operatorname{St}(F(x), \mathcal{A})$ for every $x \in U$.*

P r o o f. (a)$\Rightarrow$(b) Let $\mathcal{A}$ be an open cover of Y and $x_0 \in X$, then for each $y \in F(x_0)$ we fix $W_y \in \mathcal{A}$ with $y \in W_y$. The family $\{W_y\colon y \in F(x_0)\}$ is an open cover of $F(x_0)$. Furthermore, $F(x_0)$ is compact, we can choose $W_{y_1}, W_{y_2}, \dots, W_{y_n}$ such that $F(x_0) \subset \bigcup_{i=1}^{n} W_{y_i}$, where $n \in \mathbb{N}$. By the lower semi-continuity of F for each $i \leq n$ there exists a neighbourhood U_i of x_0 such that $F(x) \cap W_{y_i} \neq \emptyset$ for $x \in U_i$. Letting $U = \bigcap_{i=1}^{n} U_i$ we obtain $F(x) \cap W_{y_i} \neq \emptyset$ for all $x \in U$, $i \leq n$. For $y \in F(x_0)$ we have $y \in W_{y_i}$ for some $i \leq n$. Thus $y \in \operatorname{St}\bigl(F(x), \mathcal{A}\bigr)$ for $x \in U$ which give $F(x_0) \subset \operatorname{St}(F(x), \mathcal{A})$.

(b)$\Rightarrow$(a) Let $V \subset Y$ be an open set with $F(x_0) \cap V \neq \emptyset$ and let $y \in F(x_0) \cap V$. By the regularity space of Y there exists an open set $K \subset Y$ such that $y \in K \subset \overline{K} \subset V$. The family $\mathcal{A} = \{V, Y \setminus \overline{K}\}$ is an open cover. By assumption there exists a neighbourhood W of x_0 such that $F(x_0) \subset \mathrm{St}(F(x), \mathcal{A})$ for $x \in W$. Hence $y \in \mathrm{St}\big(F(x), \mathcal{A}\big)$ for $x \in W$, which implies $F(x) \cap V \neq \emptyset$ for $x \in W$, and the proof is complete. ◻

Theorem 6. *Let X be a regular space, Y — a paracompact Hausdorff one, and let $F, F_j : X \to Y$, $j \in J$ be multivalued maps. Suppose that F is continuous with compact values. If the net $\{F_j : j \in J\}$ is convergent to F in the sense of Leader with respect to $\delta^\star$ then this net convergence to F upper strongly uniformly on compact sets.*

P r o o f. Let $A \subset X$ be a compact set and let $\mathcal{G}$ be an open cover of the space Y. Since Y is a paracompact space, there exists an open cover $\mathcal{G}'$ which is a star refinement of $\mathcal{G}$. Let $p \in A$. It is known that $F(p) \subset \mathrm{St}\big(F(p), \mathcal{G}'\big)$ and F is upper semi-continuous then there exists a neighbourhood K_p' of the point p such that for every $x \in K_p'$ we have $F(x) \subset \mathrm{St}\big(F(p), \mathcal{G}'\big)$. The map F is a lower semi-continuous with the compact values, then there exists a neighbourhood K_p'' of the point p such that for every $x \in K_p''$ we have $F(p) \subset \mathrm{St}\big(F(x), \mathcal{G}'\big)$. Let $K_p = K_p' \cap K_p''$. Of course, $p \in K_p$, $F(x) \subset \mathrm{St}\big(F(p), \mathcal{G}'\big)$ and $F(p) \subset \mathrm{St}\big(F(x), \mathcal{G}'\big)$ for every $x \in K_p$. Moreover, X is a regular space, so there exists a neighbourhood U_p of the point p such that $x \in U_p \subset \overline{U}_p \subset K_p$. Thus for every $p \in A$ there exists an open neighbourhood U_p of the point p such that

$$F(x) \subset \mathrm{St}\big(F(p), \mathcal{G}'\big) \text{ and } F(p) \subset \mathrm{St}\big(F(x), \mathcal{G}'\big) \qquad \text{for every } x \in \overline{U}_p. \qquad (*)$$

In the consequence $A \subset \bigcup_{p \in A} U_p$. The set A is compact, so we can choose the points $p_1, \ldots, p_n \in A$ such that $A \subset \bigcup_{k=1}^{n} U_{p_k} \subset \bigcup_{k=1}^{n} \overline{U}_{p_k}$. Thus $A = \bigcup_{k=1}^{n} A \cap \overline{U}_{p_k}$. Because $A \cap \overline{U}_{p_k}$ is a compact set, so $F(A \cap \overline{U}_{p_k})$ is compact. Moreover, $F(A \cap \overline{U}_{p_k}) \subset \mathrm{St}\big(F(p_k), \mathcal{G}'\big)$, then $F(A \cap \overline{U}_{p_k}) \not\delta^\star \big[Y \setminus \mathrm{St}(F(p_k), \mathcal{G}')\big]$ for $k = 1, 2, \ldots, n$. From the Leader convergence there exists j_0 such that $F_j(A \cap \overline{U}_{p_k}) \not\delta^\star \big[Y \setminus \mathrm{St}(F(p_k), \mathcal{G}')\big]$. So

$$F_j(A \cap \overline{U}_{p_k}) \subset \mathrm{St}\big(F(p_k), \mathcal{G}'\big) \qquad \text{for } j \geq j_0 \text{ and } k = 1, 2, \ldots, n. \qquad (**)$$

Let $x \in A$. Then $x \in A \cap \overline{U}_{p_k}$ for some $k \leq n$. From $(**)$ we have $F_j(x) \subset \mathrm{St}\big(F(p_k), \mathcal{G}'\big)$ for $j \geq j_0$. We fix any $j \geq j_0$. Let $y \in F_j(x)$. There exists $w \in F(p_k)$ and $V \in \mathcal{G}'$ such that $y, w \in V$. Since $\mathcal{G}'$ is star refinement of $\mathcal{G}$, so there exists $W \in \mathcal{G}$ such that $\mathrm{St}(V, \mathcal{G}') \subset W$. From $(*)$ there exists $z \in F(x)$ and $U \in \mathcal{G}'$ such that $w, z \in U$. Then $z, y \in \mathrm{St}(w, \mathcal{G}') \subset \mathrm{St}(V, \mathcal{G}') \subset W$.

So $y \in \mathrm{St}(z, \mathcal{G}) \subset \mathrm{St}\big(F(x), \mathcal{G}\big)$. Since $y \in F_j(x)$, so $F_j(x) \subset \mathrm{St}\big(F(x), \mathcal{G}\big)$ for $x \in A$. This means, that the net $\{F_j : j \in J\}$ is convergent to F upper strongly uniformly on compact sets. □

COROLLARY. *Let X be a compact Hausdorff space, Y — a paracompact Hausdorff one, and let $F, F_j : X \to Y$, $j \in J$, be multivalued maps. If F is continuous with compact values then the following conditions are equivalent:*

(a) *the net $\{F_j : j \in J\}$ is convergent to F in the sense of Leader with respect to $\delta^\star$;*

(b) *the net $\{F_j : j \in J\}$ is convergent to F upper strongly uniformly.*

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