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ON COUNTABLY CONTINUOUS FUNCTIONS

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ABSTRACT. A function $f\colon \mathbb{R} \to \mathbb{R}$ is countably continuous if there are continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ such that the graph of f is contained in the union of the graphs of f_n. We prove that the family of all countably continuous functions is closed with respect to the algebraic and lattice operations and that the superposition of two countably continuous functions is countably continuous. Moreover, we show some examples of monotone function and an approximately continuous function which are not countably continuous.

1. Essential properties

Definition 1. Let $\mathbb{R}$ be the set of all reals. We will say that a function $f\colon \mathbb{R} \to \mathbb{R}$ is *countably continuous* if there is a sequence of continuous functions $f_n\colon \mathbb{R} \to \mathbb{R}$ such that

$$\mathrm{Gr}(f) \subset \bigcup_{n \ge 1} \mathrm{Gr}(f_n)\,, \tag{1}$$

where $\mathrm{Gr}(f)$ denotes the graph of the function f.

Theorem 1. *If functions $f, g\colon \mathbb{R} \to \mathbb{R}$ are countably continuous and $F\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous, then the function $x \mapsto F\big(f(x), g(x)\big)$ is also countably continuous.*

Proof. There are sequences (f_n) and (g_n) of continuous functions such that

$$\mathrm{Gr}(f) \subset \bigcup_{n \ge 1} \mathrm{Gr}(f_n) \quad \text{and} \quad \mathrm{Gr}(g) \subset \bigcup_{n \ge 1} \mathrm{Gr}(g_n)\,.$$

For $n \ge 1$ let

$$A_n = \big\{x\colon f_n(x) = f(x)\big\} \quad \text{and} \quad B_n = \big\{x : g_n(x) = g(x)\big\}\,.$$

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Then

$$\mathbb{R} = \bigcup_{n\ge 1} A_n = \bigcup_{n\ge 1} B_n \,.$$

Put

$$C_{n,k} = A_n \times B_k \quad \text{and} \quad D_{n,k} = A_n \cap B_k \qquad \text{for } n,k \ge 1\,.$$

The functions $F(f_n, g_k)$ are continuous and for every point $(x,y) \in C_{n,k}$ we have

$$F\big(f_n(x), g_k(y)\big) = F\big(f(x), g(y)\big)\,.$$

Consequently, for all points $x \in D_{n,k}$ (if $D_{n,k} \ne \emptyset$)

$$F\big(f_n(x), g_k(x)\big) = F\big(f(x), g(x)\big)\,.$$

So,

$$\operatorname{Gr}\big(F(f,g)\big) \subset \bigcup_{n,k\ge 1} \operatorname{Gr}\big(F(f_n, g_k)\big)\,,$$

and the proof is complete. □

Corollary 1. *If $a, b \in \mathbb{R}$ are constant, $\varphi\colon \mathbb{R} \to \mathbb{R}$ is a continuous function and $f, g\colon \mathbb{R} \to \mathbb{R}$ are countably continuous functions, then the functions $af + bg$, fg, $\max(f,g)$, $\min(f,g)$ and $\varphi \circ f$ are also countably continuous.*

Theorem 2. *If functions $f, g\colon \mathbb{R} \to \mathbb{R}$ are countably continuous and $g(x) \ne 0$ for all $x \in \mathbb{R}$ then the quotient f/g is also countably continuous.*

P r o o f . If $g(x) \ne 0$ for all $x \in \mathbb{R}$ and g is countably continuous function then there is a sequence (g_n) of continuous functions such that

$$g_n(x) \ne 0 \qquad \text{for all } x \in \mathbb{R}$$

and

$$\operatorname{Gr}(g) \subset \bigcup_{n\ge 1} \operatorname{Gr}(g_n)\,.$$

Then by the same way as in the proof of the last theorem we show that

$$\operatorname{Gr}\left(\frac{f}{g}\right) \subset \bigcup_{n,k\ge 1} \operatorname{Gr}\left(\frac{f_n}{g_k}\right)$$

and the proof is complete. □

Theorem 3. *If functions $f, g\colon \mathbb{R} \to \mathbb{R}$ are countably continuous, then the superposition $f \circ g$ is also countably continuous.*

P r o o f . There are sequences (f_n) and (g_n) of continuous functions such that

$$\operatorname{Gr}(f) \subset \bigcup_{n\ge 1} \operatorname{Gr}(f_n) \quad \text{and} \quad \operatorname{Gr}(g) \subset \bigcup_{n\ge 1} \operatorname{Gr}(g_n)\,.$$

For $n \geq 1$ let

$$A_n = \{x\colon f_n(x) = f(x)\} \quad \text{and} \quad B_n = \{x : g_n(x) = g(x)\}.$$

Then

$$\mathbb{R} = \bigcup_{n\geq 1} A_n = \bigcup_{n\geq 1} B_n .$$

Put

$$C_{n,k} = (g_n)^{-1}(A_k) \cap B_n \qquad \text{for } n,k \geq 1 .$$

Observe that for each point $x \in \mathbb{R}$ there is an index n_1 such that $g_{n_1}(x) = g(x)$. Next, there is an index k_1 with $g(x) \in A_{k_1}$, that is

$$f(g(x)) = f_{k_1}(g(x)) = f_{k_1}(g_{n_1}(x)) .$$

So,

$$x \in B_{n_1} \cap (g_{n_1})^{-1}(A_{k_1}) = C_{n_1,k_1} ,$$

and

$$\mathbb{R} = \bigcup_{n,k\geq 1} C_{n,k} .$$

If $x \in C_{n,k}$, then

$$f(g(x)) = f(g_n(x)) = f_k(g_n(x)) .$$

So,

$$\mathrm{Gr}(f \circ g) \subset \bigcup_{n,k\geq 1} \mathrm{Gr}(f_k \circ g_n) .$$

Since the superpositions $f_k \circ g_n$ are continuous, the proof is complete. $\square$

2. Examples

Example 1. ([2]) A monotone function $f\colon \mathbb{R} \to \mathbb{R}$ which is discontinuous on a dense set is not countably continuous.

Suppose that f is countably continuous. Then there is a sequence (f_n) of continuous functions such that (1) holds. Let

$$A_n = \{x\colon f(x) = f_n(x)\} \qquad \text{for } n \geq 1 .$$

Since

$$\mathbb{R} = \bigcup_{n\geq 1} A_n ,$$

there is an index k such that the set A_k is of the second category. So, there is an open interval I such that the intersection $I \cap A_k$ is dense in I. Since the functions f_k and f are equal on the set $A_k \cap I$ which is dense in I, they are equal at all continuity points of f belonging to I. But there is a discontinuity point $u \in I$ of the function f. Then

$$\lim_{x \to u^-} f_k(x) = \lim_{x \to u^-} f(x) \neq \lim_{x \to u^+} f(x) = \lim_{x \to u^+} f_k(x),$$

which contradicts the continuity of f_k.

Definition 2. A function $f\colon \mathbb{R} \to \mathbb{R}$ is called *additive* if it satisfies Cauchy's equation

$$f(x+y) = f(x) + f(y)$$

for all points $x, y \in \mathbb{R}$.

Remark 1. It is well-known ([4]) that the affine functions are the only continuous additive functions. However there are countably continuous discontinuous additive functions. For example every discontinuous additive function which has only countably many values is such a function ([4]).

Example 2. There are countably continuous additive discontinuous functions with uncountable images.

Let $H \subset \mathbb{R}$ be a Hamel basis of the linear space $\mathbb{R}$ over the field $\mathbb{Q}$ of all rationals. Fix a point $h_0 \in H$ and put

$$f(x) = \begin{cases} x & \text{for } x \in H \setminus \{h_0\}, \\ 0 & \text{for } x = h_0 . \end{cases}$$

Denote by Φ the family of all functions

$$g_r(x) = x - rh_0 \qquad \text{for } x \in \mathbb{R} \text{ and } r \in \mathbb{Q}.$$

Evidently the family Φ is countable and each function belonging to Φ is continuous.

The additive extension $g\colon \mathbb{R} \to \mathbb{R}$ of $f\colon H \to \mathbb{R}$ is discontinuous (see [4]) and the image $g(\mathbb{R})$ is uncountable. For each point $x \in \mathbb{R}$ there is a positive integer k and two systems $h_1, \dots, h_k \in H$ and $r_0, r_1, \dots, r_k \in \mathbb{Q}$ such that

$$x = r_0 h_0 + r_1 h_1 + \dots + r_k h_k .$$

Since the function g is additive, we obtain

$$g(x) = r_1 f(h_1) + \dots + r_k f(h_k) = r_1 h_1 + \dots + r_k h_k = x - r_0 h_0 .$$

Consequently, we have

$$\big(x, g(x)\big) \in \mathrm{Gr}(g_{r_0}) \subset \bigcup_{\psi \in \Phi} \mathrm{Gr}(\psi)$$

and

$$\mathrm{Gr}(g) \subset \bigcup_{\psi \in \Phi} \mathrm{Gr}(\psi).$$

So, the function g is countably continuous.

Definition 3. By μ denote the Lebesgue measure in $\mathbb{R}$. For a Lebesgue measurable set $A \subset \mathbb{R}$ and a point x we define the *density* $D(A, x)$ of the set A at the point x as

$$\lim_{h \to 0+} \frac{\mu\bigl(A \cap [x-h, x+h]\bigr)}{2h}\,.$$

Definition 4. A function $f\colon \mathbb{R} \to \mathbb{R}$ is called *approximately continuous* at a point x if there is a Lebesgue measurable set $A \ni x$ such that $D(A, x) = 1$ and the restricted function $f|A$ is continuous at x [1].

Example 3. There is a bounded approximately continuous function which is not countably continuous.

Let $A = \{x_n : n \geq 1\}$ be a countable set dense in $\mathbb{R}$ and let

$$g(x) = \sum_{x_n < x} \frac{1}{2^n}\,.$$

Then $g\colon \mathbb{R} \to (0,1)$ is a monotone function continuous at each point $x \notin A$ and discontinuous at all points $x \in A$. Let $B \supset A$ be a G_δ-set of measure zero. Since the function g belongs to the first Baire class (as a monotone function), by Laczkovich–Petruska theorem [5] there is an approximately continuous function $f\colon \mathbb{R} \to (0,1)$ such that $f(x) = g(x)$ for $x \in B$.

For the proof that f is not countably continuous it suffices to repeat the argument used above for monotone function.

Suppose that there is a sequence of continuous functions $f_n\colon \mathbb{R} \to \mathbb{R}$ such that (1) holds. For $n \geq 1$ let

$$A_n = \bigl\{x : f(x) = f_n(x)\bigr\}\,.$$

Since

$$\mathbb{R} = \bigcup_{n \geq 1} A_n\,,$$

there is an index k such that the set A_k is of the second category. There is an open interval I such that the set $I \cap A_k$ is residual in I. Since the set B is residual, the intersection $A_k \cap B \cap I$ is also residual in I.

Fix a point $u \in I \cap A$. Since

$$\lim_{x \to u^-} g(x) < \lim_{x \to u^+} g(x)$$

and

$$f_k|(A_k \cap B) = f|(A_k \cap B) = g|(A_k \cap B)\,,$$

we obtain a contradiction with the continuity of f_k at u.

3. Limits of sequences of countably continuous functions

Remark 2. Since all functions having countably many values are countably continuous and every function $f\colon \mathbb{R} \to \mathbb{R}$ is the limit of a uniformly converging sequence of functions with countable images, the uniform convergence does not preserve the countable continuity.

Definition 5. A sequence (f_n) of functions $f_n\colon \mathbb{R} \to \mathbb{R}$ *discretely converges* to a function f if for each point $x \in \mathbb{R}$ there is an index $n(x)$ such that $f_n(x) = f(x)$ for $n \geq n(x)$.

Theorem 4. *If a function f is the discrete limit of a sequence of countably continuous functions $f_n\colon \mathbb{R} \to \mathbb{R}$ then f is also countably continuous.*

Proof. Of course, for each positive integer n there are continuous functions $f_{n,k}\colon \mathbb{R} \to \mathbb{R}$, $k \geq 1$, such that

$$\operatorname{Gr}(f_n) \subset \bigcup_{k\geq 1} \operatorname{Gr}(f_{n,k}).$$

Since f is the discrete limit of a sequence (f_n),

$$\operatorname{Gr}(f) \subset \bigcup_{n\geq 1} \operatorname{Gr}(f_n).$$

So,

$$\operatorname{Gr}(f) \subset \bigcup_{n,k\geq 1} \operatorname{Gr}(f_{n,k}),$$

and consequently, f is countably continuous. □

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