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# CARDINAL INVARIANTS $\mathfrak{p}$, $\mathfrak{t}$ AND $\mathfrak{h}$ AND REAL FUNCTIONS

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ABSTRACT. A partial order on a family of continuous functions from a topological space $X$ into $[\omega]^\omega$ is defined as follows

$$f \subseteq^* g \iff f(x) \subseteq^* g(x) \quad \text{for any } x \in X.$$

For this order variants of cardinals $\mathfrak{p}$, $\mathfrak{t}$ and $\mathfrak{h}$ are defined and their values are estimated.

## 1. Introduction

Cardinal invariants $\mathfrak{p}$, $\mathfrak{t}$ and $\mathfrak{h}$ which characterize the Boolean algebra $\mathcal{P}(\omega)/\text{fin}$, have been studied for many years. F. Hausdorff proved the inequality $\omega_1 \leq \mathfrak{p}$ in 1936—see [6], F. Rothberger showed that $\mathfrak{p} \leq \mathfrak{t} \leq \mathfrak{b}$ in 1948—see [13]. More facts concerning $\mathfrak{p}$, $\mathfrak{t}$ and their relations to topology, can be found in the survey article of E. van Douwen [3]. Cardinal invariant $\mathfrak{h}$ was introduced much later by B. Balcar, J. Pelant and P. Simon in [1]; see also, [4], [5], [11], [14] and [15]. Let us recall the definitions of these cardinal invariants in more general case—for any partially ordered set $P$. Other basic definitions used in this article are the same as those used in the book of K. Kunen [10].

The cardinal $\mathfrak{p}(P)$ is the smallest cardinality of a centered set $A \subseteq P$ such that there is no (nonzero) $p \in P$ which is less than any member of $A$.

The cardinal $\mathfrak{t}(P)$ is the smallest cardinality of a decreasing sequence $\{a_\alpha \colon \alpha < \mathfrak{t}(P)\} \subseteq P$ such that there is no (nonzero) $p \in P$ which is less than any $a_\alpha$.

The cardinal $\mathfrak{h}(P)$ is the smallest cardinality of a family of maximal antichains such that there is no maximal antichain which refines every member of this family.

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If $P$ is a Boolean algebra, then $\mathfrak{h}(P)$ is the smallest cardinal number $\kappa$ such that the algebra is not $\kappa$-distributive. The proof of this fact can be found in the article of T. Jech [9].

The cardinal number $\mathfrak{h}^1(P)$ is the smallest cardinality of a family of maximal antichains $\Theta$ such that for any $p \in P$ there exists an antichain $A \in \Theta$ in which there are two elements compatible (consistent) with $p$. A family of maximal antichains having such property is called a *shattering family.*

For any partially ordered set the following inequalities hold

$$\mathfrak{p}(P) \leq \mathfrak{t}(P) \leq \mathfrak{h}(P) \leq \mathfrak{h}^1(P).$$

Let us recall the Base Tree Theorem.

**Base Tree Theorem.** *Let $P$ be a partially ordered set of the cardinality less than $2^\kappa$. If $\kappa < \mathfrak{t}(P)$ and $\mathfrak{h}(P) = \mathfrak{h}^1(P)$, then there exists a family of maximal antichains $\Theta = \{A_\alpha \colon \alpha < \mathfrak{h}(P)\}$ which is well ordered by the relation of refining antichains, such that for any $p \in P$ there is $q \in \bigcup \Theta$ less than $p$.*

The theorem was formulated and proved by B. Balcar, J. Pelant and P. Simon [1] for a particular case of the Boolean algebra $\mathcal{P}(\omega)/\mathrm{fin}$. A similar formulation can be found in the paper of Repický [12].

J. Hirschorn in [7], [8] investigated the values of $\mathfrak{p}, \mathfrak{t}$ under adding one Cohen or one Solovay real to a model of set theory—he examined properties of orders on a family of measurable functions and on a family of functions having the property of Baire. In this paper we extend these investigations to families of continuous functions and to a cardinal invariant $\mathfrak{h}$. The partial orders are defined as follows: The domain of functions is a topological space—usually it is a subset of the Cantor set whilst the codomain is always the family of all infinite subsets of the set of natural numbers. The codomain can be identified with a subspace of the Cantor set, since there is a correspondence between subsets of natural numbers and their characteristic functions. A function $f$ is almost contained in a function $g$ if and only if $f(x)$ is almost contained in $g(x)$ for almost all $x$.

$$f \subseteq^* g \iff f(x) \subseteq^* g(x) \qquad \text{for almost all } x.$$

(For sets $A, B$, $A \subseteq^* B$ means that $|A \setminus B| < \omega$). The phrase "for almost all $x$" has a different meaning depending on which class of functions it is considered. If the class of functions having the property of Baire is considered then "for almost all $x$" means "for all except meager set". Similarly, if the class of measurable functions is considered then "for almost all $x$" means "for all except null set", whilst, if the class of continuous functions is considered then "for almost all $x$" means just "for all $x$".

By $C(X)$ denote the family of all continuous functions from $X$ into $[\omega]^\omega$. To make the notation shorter and more visual, $\mathfrak{p}_C(X)$, $\mathfrak{t}_C(X)$, $\mathfrak{h}_C(X)$ and $\mathfrak{h}^1_C(X)$

is written instead of $\mathfrak{p}\big(C(X)\big)$, $\mathfrak{t}\big(C(X)\big)$, $\mathfrak{h}\big(C(X)\big)$ and $\mathfrak{h}^1\big(C(X)\big)$.

If a domain of functions is a discrete space, then the classical invariants $\mathfrak{p}$, $\mathfrak{t}$ are obtained. Unfortunately the same could not be said about cardinal $\mathfrak{h}$. Denote the cardinal $\mathfrak{h}_C(X)$ by $\mathfrak{h}(\kappa)$, if $X$ is a discrete space of cardinality $\kappa$. The following proposition is true.

**Proposition.** *For any cardinals $\lambda$, $\kappa$, if $\lambda \le \kappa$, then $\mathfrak{h}(\kappa) \le \mathfrak{h}(\lambda)$.*

In [1], B. Balcar, J. Pelant and P. Simon asked, whether it was possible to have $\mathfrak{h}(2) < \mathfrak{h}$. S. Shelah and O. Spinas showed in [14] and [15], that for any natural number $n$ there exists a model of set theory such that $\mathfrak{h}(n+1) < \mathfrak{h}(n)$. For any discrete space $X$ the equality $\mathfrak{h}_C(X) = \mathfrak{h}^1_C(X)$ holds and the assumptions of the Base Tree Theorem are fulfilled.

It is an easy fact that cardinals $\mathfrak{p}_C(X)$ and $\mathfrak{t}_C(X)$ are not greater than classical $\mathfrak{p}$ and $\mathfrak{t}$. In this article it is shown that they are at least $\omega_1$ and if $X$ is a countable set or a Luzin set then $\mathfrak{p}_C(X) = \mathfrak{p}$ and $\mathfrak{t}_C(X) = \mathfrak{t}$. If $X$ is the Cantor set, then $\mathfrak{p}_C(X) = \mathfrak{p}$. It is also proved that cardinal $\mathfrak{h}_C(X)$ is never greater than classical $\mathfrak{h}$, and a model of set theory is found in which the strong inequality $\mathfrak{h}_C(2^\omega) < \mathfrak{h}$ holds. Assumptions of the Base Tree Theorem are checked also.

## 2. The $\mathfrak{p}$-like and $\mathfrak{t}$-like cardinals

Technical tricks used by J. Hirschorn in [7] could be improved as follow.

Suppose that $X$ is a topological space and contains a dense and countable subset $Q \subseteq X$ such that $X$ has a countable base at each point $x \in Q$.

The following notation is used for any subset of the set of natural numbers $V$ and any natural number $n$.

$$[V|n] = \big\{A \subseteq \omega \colon A \cap \{0,1,\dots,n-1\} = V \cap \{0,1,\dots,n-1\}\big\}.$$

Sets $[V|n]$ are closed-open bases subsets of the Cantor set. Let us recall that a (transfinite) sequence $\{A_\alpha \colon \alpha < \lambda\} \subset [\omega]^\omega$ is a *tower* whenever $A_\beta \subseteq^* A_\alpha$ for any $\alpha < \beta < \lambda$.

**Lemma 2.1.** *Let $\lambda < \mathfrak{t}$ and let $f_\alpha \colon X \to [\omega]^\omega$ be continuous functions. If $\big\{f_\alpha(x) \colon \alpha < \lambda\big\}$ is a tower for any $x \in Q$, then there exists a dense $G_\delta$ subset $Y \subseteq X$ and a Borel function $f \colon Y \to [\omega]^\omega$ such that $Q \subseteq Y$ and $f(x) \subseteq^* f_\alpha(x)$ for any $\alpha < \lambda$ and each $x \in Y$.*

Proof. Let $x_0, x_1, x_2, \dots$ be an enumeration of a dense and countable subset $Q \subseteq X$, such that each point of $Q$ occurs infinitely many times in this sequence. For any $x_n$ consider a decreasing base $\big\{V_n^k \colon k < \omega\big\}$ at this point.

Each set $[f_\alpha(x_n)|k]$ is a neighbourhood of the point $f_\alpha(x_n)$. It has been assumed that any function $f_\alpha$ is continuous. So a natural numbers $\Phi_\alpha(n,k)$ could be fixed such that the image $f_\alpha(V_n^{\Phi_\alpha(n,k)})$ is always contained in $[f_\alpha(x_n)|k]$. Since $\lambda < \mathfrak{b}$ and any family $\{V_n^k : k < \omega\}$ is decreasing, a natural number $\Phi(n,k)$ could be fixed such that for any $\alpha < \lambda$

$$V_n^{\Phi(n,k)} = W(n,k) \subseteq V_n^{\Phi_\alpha(n,k)} = W_\alpha(n,k)$$

holds for all but finitely many pairs $(n,k) \in \omega \times \omega$. Therefore, for any $\alpha < \lambda$

$$f_\alpha\big(W(n,k)\big) \subseteq f_\alpha\big(W_\alpha(n,k)\big) \subseteq [f_\alpha(x_n)|k] \tag{$*$}$$

holds for all but finitely many pairs $(n,k) \in \omega \times \omega$.

It has been assumed that for any $n$ the transfinite sequence $\{f_\alpha(x_n) : \alpha < \lambda\}$ is a tower. Since $\lambda < \mathfrak{t}$, there are infinite sets of natural numbers $D_n^*$ such that $D_n^* \subseteq^* f_\alpha(x_n)$ for any $\alpha$. Since $\lambda < \mathfrak{b}$, finite subsets of each $D_n^*$ could be rejected and subsets $D_n \subseteq D_n^*$ could be obtained, such that $D_n \subseteq f_\alpha(x_n)$ for any $\alpha < \lambda$ for all but finitely many $n < \omega$.

Fix an increasing sequence of natural numbers $\{k_n : n < \omega\}$ such that $k_n \in D_n$ for any $n$. Put

$$Y = \bigcap_{n<\omega} \bigcup_{m>n} W(m, k_m + 1) \subseteq X.$$

Each point of $Q$ occurs infinitely many times in the sequence $x_0, x_1, x_2, \ldots$, hence it is concluded that $Q \subseteq Y$ and $Y$ is a $G_\delta$ dense subset of $X$. A desired function $f$ is defined by the formula

$$f(x) = \{k_n : x \in W(n, k_n + 1)\}.$$

For any natural number $n$ we have

$$f^{-1}\big(\{A \subseteq \omega : k_n \in A\}\big) = W(n, k_n + 1) \cap Y.$$

Hence the function $f$ is Borel.

For any $x \in Y$ the value $f(x)$ is infinite because of the sequence $k_0, k_1, \ldots$ is increasing and $x \in W(m, k_m + 1)$ for infinitely many $m$.

Fix an ordinal $\alpha < \lambda$ and a point $x \in Y$. Consider some $k_n \in f(x)$; it means that $x \in W(n, k_n + 1)$. By the definition of $D_n$, for all but finitely many $n$ it has to be $k_n \in D_n \subseteq f_\alpha(x_n)$. Because of $(*)$,

$$f_\alpha(x) \in f_\alpha\big(W(n, k_n + 1)\big) \subseteq [f_\alpha(x_n)|k_n + 1]$$

holds for all but finitely many $n$. If $f_\alpha(x) \in [f_\alpha(x_n)|k_n + 1]$, then

$$f_\alpha(x) \cap \{0, 1, \ldots, k_n\} = f_\alpha(x_n) \cap \{0, 1, \ldots, k_n\}.$$

This implies that $k_n \in f_\alpha(x_n)$ if, and only if $k_n \in f_\alpha(x)$. Therefore, for all but finitely many $n$ it has to be $k_n \in f_\alpha(x)$. Thus it was shown that $f(x) \subseteq^* f_\alpha(x)$ for any $\alpha < \lambda$ and each $x \in Y$. □

The function $f$ could be extended by the same formula on the whole space $X$ except for a meager set $X \setminus Y$ where it has finite values. Moreover, $f$ is continuous if sets $W(n,k)$ are closed-open.

**THEOREM 2.2.** *Assume that $X$ is a zero-dimensional. Let $\lambda < \mathfrak{t}$ and let $f_\alpha \colon X \to [\omega]^\omega$ be continuous functions for any $\alpha < \lambda$. If $\{f_\alpha(x) \colon \alpha < \lambda\}$ is a tower for any $x \in Q$, then there exists a dense $G_\delta$ subset $Y \subseteq X$ and a continuous function $f \colon Y \to [\omega]^\omega$ such that $Q \subseteq Y$ and $f(x) \subseteq^* f_\alpha(x)$ for any $\alpha < \lambda$ and each $x \in Y$.*

P r o o f. Let us consider the above proof, but assume that the sets $V_n^k$ are closed and open. Then the preimages

$$f^{-1}\big(\{A \subseteq \omega \colon k_n \in A\}\big) = W(n, k_n + 1),$$

are closed-open and $f$ is continuous. □

A subset of the Cantor set is a *Luzin set* whenever it is uncountable and any its meager subset is countable. Notice that the assumptions of the theorem below are strengthened!

**THEOREM 2.3.** *Let $X \subseteq 2^\omega$ be a countable set or a Luzin set and $f_\alpha \colon X \to [\omega]^\omega$ be continuous functions such that $\{f_\alpha(x) \colon \alpha < \lambda\}$ is a tower for any $x \in X$. There exists a continuous function $f \colon X \to [\omega]^\omega$ such that $f(x) \subseteq^* f_\alpha(x)$ for any $\alpha < \lambda$ and each $x \in X$.*

P r o o f. Lemma 2.1 is used twice. For the first time a continuous function $f' \colon X \to \mathcal{P}(\omega)$, such that $f'(x) \subseteq^* f_\alpha(x)$ for any $\alpha < \lambda$ and each $x \in X$ is obtained. The function $f'$ has finite values on a meager set $Q$. Since $X$ is a Luzin set, $Q$ has to be countable. Increase $Q$ to a dense countable subset of $X$ and use Theorem 2.2. Another function $f'' \colon X \to \mathcal{P}(\omega)$ will be obtained. Now, it is sufficient to put $f(x) = f'(x) \cup f''(x)$ for any $x \in X$. □

Note that if the cardinal $\mathfrak{p}$ instead of $\mathfrak{t}$ is considered in the above theorems and if families $\{f_\alpha(x) \colon \alpha < \lambda\}$ are assumed to be centered, then a similar result will be obtained. Let us recall that a family $\{A_\alpha \colon \alpha < \lambda\} \subset [\omega]^\omega$ is a *centered family* whenever for any finite subset $F \subset \lambda$ there exists an infinite set $A \in [\omega]^\omega$ such that $A \subseteq^* A_\alpha$ for any $\alpha \in F$.

The theorems above can be expressed as follows.

**THEOREM 2.4.** *Whenever $X$ is a countably set or Luzin set, then $\mathfrak{p}_C(X) \geq \mathfrak{p}$ and $\mathfrak{t}_C(X) \geq \mathfrak{t}$.*

In the following theorem no special assumptions are necessary. In the previous version of this theorem it was assumed that $X$ is a compact space, but at a seminar held in Katowice A. Kucia has noticed that such assumption is not necessary.

**THEOREM 2.5.** *For any topological space $X$, $\omega_1 \leq \mathfrak{p}_C(X) \leq \mathfrak{t}_C(X) \leq \mathfrak{t}$ and $\mathfrak{p}_C(X) \leq \mathfrak{p}$ is true.*

Proof. The inequalities $\mathfrak{p}_C(X) \leq \mathfrak{t}_C(X) \leq \mathfrak{t}$ and $\mathfrak{p}_C(X) \leq \mathfrak{p}$ are trivial.

Let $\{f_n \colon n \in \omega\}$ be a decreasing sequence of continuous functions. Put $p_0(x) = \min f_0(x)$ for any $x \in X$. Assume that $p_{n-1}(x)$ has been defined. Put

$$p_n(x) = \min f_0(x) \cap f_1(x) \cap \cdots \cap f_n(x) \setminus \{p_0(x), p_1(x), \dots p_{n-1}(x)\}.$$

Notice that the defined functions $p_n \colon X \to \omega$ are continuous (the topology on $\omega$ is discrete). Define the function $f \colon X \to [\omega]^\omega$ by the formula

$$f(x) = \{p_0(x), p_1(x), p_2(x), \dots\}.$$

The function $f$ is continuous because the preimages of subbase sets $\{V \subseteq \omega \colon n \in V\}$ are finite unions of closed-open sets. We have

$$f^{-1}\big(\{V \subseteq \omega \colon n \in V\}\big) = \{x \in X \colon p_0(x) = n\} \cup \{x \in X \colon p_1(x) = n\} \cup \dots \\ \cdots \cup \{x \in X \colon p_n(x) = n\}.$$

Let $\mathbf{S} = \{s_0, s_1, \dots\}$ be the family of all simple, continuous functions from the Cantor set into the family of all nonempty finite subsets of the set of natural numbers. Let

$$\|s\| = \max\Big\{k < \omega \colon k \in \bigcup_{x \in 2^\omega} s(x)\Big\}$$

for any simple function $s \colon 2^\omega \to [\omega]^{<\omega} \setminus \{\emptyset\}$. Let us introduce the following notion: A function $f$ is contained in a function $g$ if and only if $f(x)$ is contained in $g(x)$ for any $x \in 2^\omega$. We write

$$f \subseteq g \iff f(x) \subseteq g(x) \qquad \text{for any } x \in 2^\omega.$$

For any function $f \in C(2^\omega)$ we define the sets $F^n$ by the formula:

$$F^n = \big\{k \colon (\forall x \in 2^\omega)\ s_k(x) \cap \{0, 1, \dots, n\} = \emptyset \text{ and } s_k \subseteq f\big\}. \qquad \square$$

**PROPOSITION 2.6.** *For any function $f \in C(2^\omega)$, $F^0 \supseteq F^1 \supseteq F^2 \supseteq \dots$*

**Lemma 2.7.** *Sets $F^n$ are infinite.*

Proof. Fix $f \in C(2^\omega)$. Put $U_k = \{x \in 2^\omega : k \in f(x)\}$. For any $m < \omega$ the family of sets $\{U_k : k > m\}$ is an open covering of the Cantor set, consisting of closed-open sets. Choose a finite covering $\{U_t : t \in T\}$. Define the function $s \in S$ by the formula $s(x) = \{t \in T : x \in U_t\}$. The function $s = s_{q_m}$ belongs to $\mathbf{S}$ and $q_m \in F^m$.

Fix $n < \omega$. The sequence $q_n, q_{n+1}, q_{n+2}, \dots$ has infinite many elements because of $m < \|s_{q_m}\|$. Hence by previous proposition $F^n$ is infinite. □

**Lemma 2.8.** *If $\{f_\alpha : \alpha < \lambda\} \subset C(2^\omega)$ is a centered family then $\{F^n_\alpha : \alpha < \lambda,\, n < \omega\} \subset [\omega]^\omega$ is a centered family.*

Proof. Fix $A \in [\lambda]^{<\omega}$ and $B \in [\omega]^{<\omega}$. Let $n = \max B$ and

$$f(x) = \bigcap_{\alpha \in A} f_\alpha(x) \qquad \text{for any } x \in X\,.$$

Since the set $A$ is finite, then $f\colon 2^\omega \to [\omega]^\omega$ is continuous. By the definition and Proposition 2.6 we get

$$F^n \subseteq \bigcap\{F^n_\alpha : \alpha \in A\} \subseteq \bigcap\{F^m_\alpha : \alpha \in A \text{ and } m \in B\}\,. \qquad □$$

**Lemma 2.9.** *If $\lambda < \mathfrak{p}$ and $\{f_\alpha : \alpha < \lambda\} \subset C(2^\omega)$ is a centered family, then there exists a function $f \in C(2^\omega)$ such that $f \subseteq^* f_\alpha$ for any $\alpha < \lambda$.*

Proof. The family $\{F^n_\alpha : \alpha < \lambda \text{ and } n < \omega\}$ is centered. Therefore there are infinite sets $G_n \subseteq F^n_0$ such that

$$G_n \subseteq^* F^n_\alpha$$

for any $\alpha < \lambda$ and $n \in \omega$. For any $\alpha$ let $a_\alpha \in \omega^\omega$ be a sequence such that

$$G_n \setminus \{0, 1, 2, \dots, a_\alpha(n)\} \subseteq F^n_\alpha\,.$$

Since $\lambda < \mathfrak{b}$, there exists a sequence $a \in \omega^\omega$ which almost dominates sequences $a_\alpha$. Let $H_n = G_n \setminus \{0, 1, 2, \dots, a(n)\}$. For any $\alpha < \lambda$, for almost all $n \in \omega$ the inclusion $H_n \subseteq F^n_\alpha$ holds, while for other finitely many $n$ $H_n \subseteq^* F^n_\alpha$ holds.

Let $n_0 = 0$ and $i_0 \in H_{n_0}$. Assume that $i_0, i_1, i_2, \dots, i_{k-1}$ and $n_0, n_1, n_2, \dots, n_{k-1}$ are defined. Let $n_k = \|s_{i_{k-1}}\|$ and $i_k \in H_{n_k}$. The function $f$ is defined by the formula

$$f(x) = \bigcup_{k<\omega} s_{i_k}(x) \qquad \text{for } x \in 2^\omega.$$

Since $i_k \in H_{n_k} \subseteq F^{n_k}_0$, then $s_{i_k}(x) \cap \{0, 1, \dots, n_k\} = \emptyset$ for any $x \in 2^\omega$. Hence $\|s_{i_k}\| > n_k = \|s_{i_{k-1}}\|$. Thus the sequence $n_0 < n_1 < \dots$ is strictly increasing.

If $\alpha < \lambda$ is fixed, then $i_k \in H_{n_k} \subseteq F_\alpha^{n_k}$ for almost all $k < \omega$. Thus $s_{i_k} \subseteq f_\alpha$ for almost all $k < \omega$. This implies that $f \subseteq^* f_\alpha$.

The intersection $s_{i_n}(x) \cap \{0, 1, \dots, n_k\}$ is empty for any $x \in 2^\omega$, because $i_k \in F_0^{n_k}$. Since $n_0 < n_1 < \dots$, then $f(x)$ has to be infinite for any $x \in 2^\omega$.

It remains to show that $f$ is continuous. Preimages of subbase sets are closed-open. By the definition

$$f^{-1}\big(\{A \subseteq \omega \colon m \in A\}\big) = \bigcup_{k<\omega} s_{i_k}^{-1}\big(\{A \subseteq \omega \colon m \in A\}\big).$$

Let us observe that the union on the right side of the equality is a finite union of closed-open sets, because if $n_k > m$, then $s_{i_k}(x) \cap \{0, 1, \dots, m, \dots n_k\} = \emptyset$ for any $x \in 2^\omega$, and $s_{i_k}^{-1}\big(\{A \subseteq \omega \colon m \in A\}\big) = \emptyset$. $\square$

**Theorem 2.10.** $\mathfrak{p}_C(2^\omega) = \mathfrak{p}$.

## 3. The $\mathfrak{h}$-like cardinals

In this section the Base Tree Theorem will be shown for partially ordered sets $C(X)$, and the $\mathfrak{h}$-type numbers will be estimated. First, it will be proved that the considered orders are homogeneous.

If $A \in [\omega]^\omega$ then $e_A \colon A \to \omega$ denotes the increasing bijection. Hence the sentence $n \in e_A[B]$ where $B \subseteq \omega$ means that the $n$th member of $A$ belongs to $B$. If $f \colon X \to [\omega]^\omega$ is a fixed function then for any function $g \colon X \to [\omega]^\omega$ the function $F(g) \colon X \to [\omega]^\omega$ is defined by the formula

$$F(g)(x) = e_{f(x)}\big[g(x)\big].$$

Define also a function $F^{-1}(g)$ by the formula

$$F^{-1}(g)(x) = e_{f(x)}^{-1}\big[g(x)\big].$$

**Proposition 3.1.**

(a) *If $f(x) \cap g(x)$ is an infinite set, then $F(g)(x)$ is infinite.*

(b) *$F(f)(x) = \omega$ for any $x \in X$.*

(c) *For any $x \in X$ the following equivalence holds:*

$$g(x) \cap f(x) \subseteq^* h(x) \cap f(x) \iff F(g)(x) \subseteq^* F(h)(x).$$

The function $F^{-1}$ has similar properties.

**Lemma 3.2.** *Let $f \colon X \to [\omega]^\omega$ be a continuous function.*

(a) *If a function $g \colon X \to [\omega]^\omega$ is continuous, then $F(g)$ is continuous.*

(b) *If a function $g \colon X \to [\omega]^\omega$ is continuous, then $F^{-1}(g)$ is continuous.*

Proof. (a) According to the definition we have

$$\begin{aligned} F(g)^{-1}\big(\{A \subseteq \omega : n \in A\}\big) &= \big\{x \in X : n \in F(g)(x)\big\} \\ &= \big\{x \in X : n \in e_{f(x)}\big[g(x)\big]\big\} = \big\{x \in X : e^{-1}_{f(x)}(n) \in g(x)\big\} \\ &= \bigcup_{k_1 < k_2 \leq \dots < k_n} \{x \in X : f(x) \cap \{0,1,\dots,k_n\} = \{k_0,k_1,\dots,k_n\}\} \\ &\qquad \cap \big\{x \in X : k_n \in g(x)\big\}\,. \end{aligned}$$

The preimages $F(g)^{-1}\big(\{A \subseteq \omega : n \in A\}\big)$ are open as a union of open sets Similarly it could be check that preimages $F(g)^{-1}\big(\{A \subseteq \omega : n \notin A\}\big)$ are open.

(b) The proof is similar. □

**Theorem 3.3.** *If $f \in C(X)$, then partially ordered sets $C(X)$ and $\{g \in C(X) : g \subseteq f\}$ are isomorphic.*

Proof. The function $F : \{g \in C(X) : g \subseteq f\} \to C(X)$ is an isomorphism. The inverse function is $F^{-1} : C(X) \to \big\{g \in C(X) : g \subseteq f\big\}$:

$$\big(F^{-1} \circ F\big)(g)(x) = e^{-1}_{f(x)}\big[F(g)(x)\big] = e^{-1}_{f(x)}\Big[e_{f(x)}\big[g(x)\big]\Big] = g(x)\,.$$

Similarly $\big(F \circ F^{-1}\big)(g)(x) = e_{f(x)}\big[F(g)(x)\big] = e_{f(x)}\Big[e^{-1}_{f(x)}\big[g(x)\big]\Big] = g(x)$. □

Theorem 3.3 is a tool for repeating—following B. Balcar, J. Pelant and P. Simon—the proof of the equality $\mathfrak{h}_C(X) = \mathfrak{h}^1_C(X)$ and the proof of the Base Tree Theorem.

**Theorem 3.4.** $\mathfrak{h}_C(X) = \mathfrak{h}^1_C(X)$.

**Theorem 3.5.** (Base Tree Theorem for $C(X)$). *For any topological space $X$ there exists in $C(X)$ a family of maximal antichains $\Theta = \big\{A_\alpha : \alpha < \mathfrak{h}_C(X)\big\}$ which is well ordered by the relation of refining antichains, such that for any functions $f \in C(X)$ there is in $\bigcup\Theta$ a function $g \subseteq^* f$.*

Proof. It is sufficient to observe that the assumptions of the Base Tree Theorem are fulfilled. In fact, $\big|C(X)\big| = \mathfrak{c}$, and $\omega < \mathfrak{t}_C(X)$ and $\mathfrak{h}_C(X) = \mathfrak{h}^1_C(X)$. □

Note that if the partially orders on the family of measurable functions or functions having the property of Baire instead of $C(X)$ are considered in Theorem 3.3, Theorem 3.4 and Theorem 3.5, then similar results are obtained.

It is clear that $\mathfrak{p}_C(X) \leq \mathfrak{p}$ and $\mathfrak{t}_C(X) \leq \mathfrak{t}$ hold for any topological space $X$. Similarly $\mathfrak{h}_C(X) \leq \mathfrak{h}$, but this fact is more difficult. At a seminar held in Prague B. Balcar, T. Jech and P. Simon advised to use Boolean algebras to prove this fact.

**Lemma 3.6.** *Let $X$ be a topological space. Assume that $f \in C(X)$, $A \subseteq \omega$ and $x_0 \in X$ are such that $f(x_0) \cap A$ is infinite. There exists a function $g_{f,A} \in C(X)$ such that $g_{f,A} \subseteq^* f$ and $g_{f,A}(x_0) = f(x_0) \cap A$.*

Proof. The function $g_{f,A} \colon X \to [\omega]^\omega$ is defined by the formula $g_{f,A}(x) = e^{-1}_{f(x)}\Big[e_{f(x_0)}\big[f(x_0) \cap A\big]\Big]$. □

**Lemma 3.7.** *The order $\subseteq^*$ is separative for any topological space $X$.*

Proof. Assume that $f \not\subseteq^* g$. There is a point $x_0 \in X$ such that $f(x_0) \not\subseteq^* g(x_0)$. Put $A = f(x_0) \backslash g(x_0)$ and use Lemma 3.6. The obtained function $g_{A,f} \subseteq^* f$ is incompatible with $g$. □

Since every partially ordered set $C(X)$ is separative, it can be densely embedded into a complete Boolean algebra $B(X)$.

**Lemma 3.8.** *Let $X$ be a topological space and $Y \subseteq X$. If every continuous function $f \colon Y \to [\omega]^\omega$ has a continuous extension $\bar{f} \colon X \to [\omega]^\omega$, then $\mathfrak{h}_C(X) \leq \mathfrak{h}_C(Y)$.*

Proof. In this proof it is used the fact that $\mathfrak{h}\big(C(X)\big) = \mathfrak{h}\big(B(X)\big)$. For any maximal antichain $\mathcal{A}$ in $C(Y)$ define a maximal antichain $\mathcal{H}$ in Boolean algebra $B(X)$ as follows. For any function $f \in C(X)$ choose a function $h_f \in \mathcal{A}$ such that the intersection $f(x) \cap h_f(x)$ is infinite for every $x \in Y$. There exists a continuous function $g_f = g_{f,h_f}$ such that $g_f \subseteq^* f$ and $g_f(x) = f(x) \cap h_f(x)$ for any $x \in Y$. Let

$$b_h = \sup\{g_f \colon f \in C(X),\ h_f = h\}$$

for any $h \in \mathcal{A}$, where supremum is calculated in the algebra $B(X)$. Put $\mathcal{H} = \{b_h \colon h \in \mathcal{A}\}$.

The family $\mathcal{H}$ is an antichain. Fix $h, k \in \mathcal{A}$. For any two functions $f_0, f_1$ such that $h_{f_0} = h$ and $h_{f_1} = k$ the product $g_{f_0}|Y \cap g_{f_1}|Y = h \cap k = 0$ in algebra $B(Y)$. Hence $g_{f_0} \cap g_{f_1} = 0$ in the algebra $B(X)$. Therefore $b_h \cap b_k = 0$ in $B(X)$.

The antichain $\mathcal{H}$ is maximal. Let $f \in C(X)$. Since $g_f \subseteq^* f$, then $g_f \cap f \neq 0$ in $B(X)$. Moreover, $g_f \leq b_{h_f}$ in $B(X)$. Hence $b_{h_f} \cap f \neq 0$.

Let $\{\mathcal{A}_\alpha \colon \alpha < \mathfrak{h}\}$ be a shattering family in $C(Y)$. Choose the maximal antichain $\mathcal{H}_\alpha$ in $B(X)$ for any $\mathcal{A}_\alpha$. The family $\{\mathcal{H}_\alpha \colon \alpha < \mathfrak{h}\}$ is a shattering family in $B(X)$. Fix $f \in C(X)$. There are $\alpha < \mathfrak{h}$ and $h, k \in \mathcal{A}_\alpha$ such that $f|Y \cap h \neq 0$ and $f|Y \cap k \neq 0$ in $B(Y)$. There are functions $g_{f,h}, g_{f,k} \in C(X)$ such that $g_{f,h}|Y = f|Y \cap h$ and $g_{f,k}|Y = f|Y \cap k$ and $g_{f,h}, g_{f,k} \subseteq^* f$. Of course, $f \cap b_h \neq 0$ and $f \cap b_k \neq 0$ in $B(X)$. □

**Theorem 3.9.** *For any topological space $X$, $\mathfrak{h}_C(X) \leq \mathfrak{h}$.*

P r o o f. Let $Y \subseteq X$ be a one-point set. Then $\mathfrak{h}_C(Y) = \mathfrak{h}$. It is sufficient to use Lemma 3.8—property of extending functions secure Lemma 3.6. □

**Lemma 3.10.** *If $X$ is a countable space with one limit point, $\mathfrak{h}_C(X) = \mathfrak{h}(\omega)$.*

P r o o f. If $f\colon X \to [\omega]^\omega$ is a function, then there is a continuous function $g\colon X \to [\omega]^\omega$ such that $f(x) =^* g(x)$ for any $x \in X$. Let $x_\infty \in X$ be a limit point. Enumerate $x_0, x_1, x_2, \dots$ a set $X \setminus \{x_\infty\}$. Function $g$ is defined by formulas

$$g(x_n) = \big(f(x_n) \setminus n\big) \cup \big(f(x_\infty) \cap n\big)$$

and $g(x_\infty) = f(x_\infty)$. □

**Theorem 3.11.** *We have $\mathfrak{h}_C(2^\omega) \leq \mathfrak{h}(\omega)$.*

P r o o f. Let $C_0 = \big\{x \in 2^\omega \colon x(0) = 0\big\}$ and

$$C_n = \big\{x \in 2^\omega \colon x(0) = x(1) = \dots = x(n-1) = 1 \text{ and } x(n) = 0\big\}.$$

The sets $C_n$ are closed-open, pairwise disjoint and homeomorphic to the Cantor set. Choose one point $x_n$ from each set $C_n$, and in addition put $x_\infty(n) = 1$ for any $n < \omega$. Let $Y = \{x_n \colon n < \omega\} \cup \{x_\infty\}$. The space $Y$ has one limit point $x_\infty$. Notice that

$$2^\omega = \bigcup_{n<\omega} C_n \cup \{x_\infty\}.$$

Let $f\colon Y \to [\omega]^\omega$. It could be assumed that $f$ is continuous because of Lemma 3.10. By Lemma 3.6, there exist continuous functions $g^n\colon C_n \to [\omega]^\omega$ such that $g^n(x_n) =^* f(x_n)$.

The function $\bar{f}\colon 2^\omega \to [\omega]^\omega$ is defined by formulas:

$$\bar{f}(x) = \big(g^n(x) \setminus n\big) \cup \big(f(x_\infty) \cap n\big) \qquad \text{for } x \in C_n,$$

and $\bar{f}(x_\infty) = f(x_\infty)$. The function $\bar{f}$ is a continuous extension of the function $f$. It is sufficient to use Lemma 3.8. □

In [14] and [15], S. S h e l a h and O. S p i n a s described a model such that $\omega_1 = \mathfrak{h}(n+1) < \mathfrak{h}(n) = \omega_2$, for any $n < \omega$. Therefore and by the Proposition the following corollary is obtained.

**Corollary 3.12.** *There exists a model of ZFC such that $\mathfrak{h}_C(2^\omega) < \mathfrak{h}$.*

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