Tatra Mt. Math. Publ. **28** (2004), 141–151

TATRA MOUNTAINS
Mathematical Publications

NOTE ON THE SUP-MEASURABILITY OF MULTIVALUED FUNCTIONS

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ABSTRACT. For a multivalued function of two variables there is established a possibility of reinforcement of the conditions of measurability concerning the first variable and the conditions of semicontinuity concerning the second one that causes F to be sup-measurable.

1. Introduction

For multivalued function of two variables besides of measurability (more precisely product-measurability or joint measurability), a superpositional measurability (or shortly sup-measurability) can be considered. If $F\colon X \times Y \to Z$ is a multivalued function and a superposition of Carathéodory type $F\bigl(x, G(x)\bigr)$ is measurable for every measurable multivalued function $G\colon X \to Y$, then F is called sup-measurable.

It is known (see [Cal], [Pap] and [Zy1]) that under suitable assumptions on the spaces X, Y and Z, a multivalued function $F\colon X \times Y \to Z$ with measurable y-sections and continuous x-sections (called Carathéodory multivalued function) is sup-measurable. A multivalued function F having lower (resp. upper) semicontinuous x-sections and measurable y-sections (called lower (resp. upper) semi-Carathéodory) may not be sup-measurable. Some possibilities of reinforcement of the condition that F is a semi-Carathéodory multivalued function causing F to be sup-measurable are given in [Pap] and [Zy2]. In this note we show that a lower semi-Carathéodory multivalued function F with upper quasicontinuous x-sections is sup-measurable.

We give this theorem for multivalued functions in the possible general abstract spaces.

2000 Mathematics Subject Classification: 54C60, 54C08, 28B20.
Keywords: multivalued functions, semi-Carathéodory multivalued functions, product measurability, superpositional measurability.

This research was supported by the University of Gdańsk, grant BW Nr 5100-5-0151-0.

2. Preliminaries

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{N}$ the set of positive integers and $\mathcal{L}(\mathbb{R})$ the family of all Lebesgue measurable subsets of $\mathbb{R}$.

Let T and Z be two arbitrary nonempty sets and let $\Phi\colon T \to Z$ be a multivalued function, i.e., Φ denotes a mapping such that $\Phi(t)$ is a nonempty subset of Z for $t \in T$.

The set $\mathrm{Gr}(\Phi) \subset T \times Z$ defined by the formula

$$\mathrm{Gr}(\Phi) = \{(t,z) \in T \times Z \colon z \in \Phi(t)\}$$

is called a graph of Φ.

Any function $\phi\colon T \to Z$ such that $\phi(t) \in \Phi(t)$ for each $t \in T$ is called a selection of Φ.

If $\Phi\colon T \to Z$ is a multivalued function, then two inverse images of a subset $G \subset Z$ may be defined as

$$\Phi^+(G) = \{t \in T \colon F(t) \subset G\} \quad \text{and} \quad \Phi^-(G) = \{t \in T \colon F(t) \cap G \neq \emptyset\}.$$

The following relations between these inverse images hold:

$$\Phi^-(G) = T \setminus \Phi^+(Z \setminus G) \quad \text{and} \quad \Phi^+(G) = T \setminus \Phi^-(Z \setminus G). \tag{1}$$

We also define the image of a subset $U \subset T$ under Φ by the formula

$$\Phi(U) = \bigcup_{u \in U} \Phi(u). \tag{2}$$

Let $(T, \mathcal{T}(T))$ be a topological space. The closure of a set $A \subset T$ will be denoted by $\mathrm{Cl}(A)$.

Definition 1. ([Lev]) A set $A \subset T$ is called *semi-open* if there is an open set $V \subset T$ such that $V \subset A \subset \mathrm{Cl}(V)$.

Let $(T, \mathcal{T}(T))$ and $(Z, \mathcal{T}(Z))$ be two topological spaces.

Definition 2. A multivalued function $\Phi\colon T \to Z$ is called *lower* (resp. *upper*) *semicontinuous at a point* $t_0 \in T$ if for each $V \in \mathcal{T}(Z)$ such that $\Phi(t_0) \cap V \neq \emptyset$ (resp. $\Phi(t_0) \subset V$) there exists an open neighbourhood U of the point t_0 such that $\Phi(t) \cap V \neq \emptyset$ (resp. $\Phi(t) \subset V$) for each $t \in U$.

If a multivalued function Φ is simultaneously lower and upper semicontinuous at a point $t_0 \in T$, then Φ is said to be *continuous at* t_0.

A multivalued function Φ being lower (resp. upper) semicontinuous at each point $t \in T$ is said to be *lower* (resp. *upper*) *semicontinuous.*

It is clear that a multivalued function Φ is lower (resp. upper) semicontinuous if and only if $\Phi^-(G) \in \mathcal{T}(T)$ (resp. $\Phi^+(G) \in \mathcal{T}(T)$) whenever $G \in \mathcal{T}(Z)$.

Definition 3. ([N-N]) A multivalued function $\Phi\colon T \to Z$ is said to be *lower quasicontinuous* (resp. *upper quasicontinuous*) *at a point* $t_0 \in T$ if for any open set $V \subset T$ such that $\Phi(t_0) \cap V \neq \emptyset$ (resp. $\Phi(t_0) \subset V$) and any open U containing t_0 there exists an open set $W \subset V$, $W \neq \emptyset$ such that $\Phi(t) \cap V \neq \emptyset$ (resp. $\Phi(t) \subset V$) for each $t \in W$.

A multivalued function Φ is said to be *lower* (resp. *upper*) *quasicontinuous* if it is lower (resp. upper) quasicontinuous at every point $t \in T$.

The notion lower semiquasicontinuous and upper semiquasicontinuous would be more appropriate. It would correspond to lower semicontinuous and upper semicontinuous, respectively. We omit the word “semi” for the sake of shortening.

Proposition 1. ([N-N, Theorem 2]) *Let $\big(T, \mathcal{T}(T)\big)$ be a first countable Hausdorff space and $\big(Z, \mathcal{T}(Z)\big)$ a second countable topological one. Let $\Phi\colon T \to Z$ be a compact-valued multivalued function. Then Φ is upper quasicontinuous at a point $t_0 \in T$ if and only if there exists a quasi-open set A containing t_0 such that the restriction $\Phi|A$ is upper semicontinuous at t_0.*

Let $\big(T, \mathcal{A}(T)\big)$ be a measurable space, where $\mathcal{A}(T)$ stands for some σ-algebra of subsets of T and let $\big(Z, \mathcal{T}(Z)\big)$ be a topological space.

Definition 4. A multivalued function $\Phi\colon T \to Z$ is called $\mathcal{A}(T)$-*measurable* (resp. *weakly* $\mathcal{A}(T)$-*measurable*) if the inverse image $\Phi^{-}(G)$ belongs to $\mathcal{A}(T)$ for every closed (resp. open) set $G \subset Z$.

Proposition 2. ([K-S, Proposition 1]) *Let $\big(T, \mathcal{A}(T)\big)$ be a measurable space and let $\big(Z, \mathcal{T}(Z)\big)$ be a topological one. Then*

(i) *If $\big(Z, \mathcal{T}(Z)\big)$ is a perfect space and Φ is $\mathcal{A}(T)$-measurable, then Φ is weakly $\mathcal{A}(T)$-measurable.*

(ii) *If $\big(Z, \mathcal{T}(Z)\big)$ is a perfectly normal and Φ is compact-valued, then Φ is $\mathcal{A}(T)$-measurable if and only if Φ is weakly $\mathcal{A}(T)$-measurable.*

Proposition 3. *Let $\big(T, \mathcal{A}(T)\big)$ be a measurable space, (Z, ρ) a metric space and $\Phi\colon T \to Z$ a multivalued function with closed values. For a fixed $z \in Z$ let us define a real function $g_z\colon T \to \mathbb{R}$ by the formula*

$$g_z(t) = \rho\big(z, \Phi(t)\big)\,.$$

(i) ([Him], Theorem 3.3) *Consider the statements:*

(a) *Φ is weakly $\mathcal{A}(T)$-measurable;*

(b) *for each $z \in Z$ the real function g_z is $\mathcal{A}(T)$-measurable;*

(c) $\mathrm{Gr}(\Phi) \in \mathcal{A}(T) \otimes \mathcal{B}(Z)$, *where $\mathcal{A}(T) \otimes \mathcal{B}(Z)$ stands for the σ-algebra generated by the family of sets $A \times B$ whenever $A \in \mathcal{A}(T)$ and B is* *a Borel subset of* Z.

If (Z,ρ) *is separable, then* (a) $\iff$ (b) $\implies$ (c).

(ii) ([C-V], Theorem III. 30) *If* (Z,ρ) *is complete and separable, then* Φ *is weakly* $\mathcal{A}(T)$*-measurable if and only if* Φ *admits a sequence of* $\mathcal{A}(T)$*-measurable selections* $(\phi_n)_{n\in\mathbb{N}}$ *such that* $\Phi(t)=\mathrm{Cl}\{\phi_n(t)\colon n\in\mathbb{N}\}$ *for any* $t\in T$.

Definition 5. ([Frm], Definition 1) A measurable space *with negligibles* is a triple $\big(T,\mathcal{A}(T),\mathcal{I}(T)\big)$, where T is a set, $\mathcal{A}(T)$ is a σ-algebra of subsets of T, and $\mathcal{I}(T)\subset\mathcal{P}(T)$ is a σ-ideal of the Boolean algebra $\mathcal{P}(T)$ generated by $\mathcal{I}(T)\cap\mathcal{A}(T)$.

This concept of a measurable space with negligibles is a common generalization of the two principal examples: the space $\big(T,\mathcal{A}(T),\mathcal{N}(T)\big)$, where $\big(T,\mathcal{A}(T),\mu\big)$ is a measure space and $\mathcal{N}(T)$ is σ-ideal of μ-measure zero subsets of T, and the space $\big(T,\widehat{\mathcal{B}}(T),\mathcal{M}(T)\big)$, where $\big(T,\mathcal{T}(T)\big)$ is a topological space, $\widehat{\mathcal{B}}(T)$ is σ-algebra of subsets of T with the Baire property, and $\mathcal{M}(T)$ is σ-ideal of meager subsets of T.

Definition 6. ([Frm], Definition 1B) Let $\big(T,\mathcal{A}(T),\mathcal{I}(T)\big)$ be a measurable space with negligibles and κ a cardinal.

(a) $\big(T,\mathcal{A}(T),\mathcal{I}(T)\big)$ is said to be *complete* if $\mathcal{I}(T)\subset\mathcal{A}(T)$.

(b) $\big(T,\mathcal{A}(T),\mathcal{I}(T)\big)$ is *κ-saturated* if whenever $\mathcal{E}\subset\mathcal{A}(T)\setminus\mathcal{I}(T)$ and cardinality of $\mathcal{E}$ is κ there are distinct E and F in $\mathcal{E}$ such that $E\cap F\notin\mathcal{I}(T)$.

(c) $\big(T,\mathcal{A}(T),\mathcal{I}(T)\big)$ is *proto-decomposable* if there is a family $\mathcal{U}\subset\mathcal{A}(T)$ such that whenever $\mathcal{W}\subset\mathcal{U}$ is disjoint, then

(i) $\bigcup\mathcal{W}\in\mathcal{A}(T)$,

(ii) if $\{E_W\}_{W\in\mathcal{W}}$ is a family in $\mathcal{I}(T)$, then $\bigcup\limits_{W\in\mathcal{W}}(W\cap E_W)\in\mathcal{I}(T)$,

(iii) if $E\subset T$ and $E\setminus\bigcup\mathcal{W}\in\mathcal{A}(T)\setminus\mathcal{I}(T)$, then there is a non-empty set $U\in\mathcal{U}$ such that $U\cap\bigcup\mathcal{W}=\emptyset$ and $U\setminus E\in\mathcal{I}(T)$.

Among examples of proto-decomposable spaces there are $\big(T,\mathcal{A}(T),\mathcal{N}(T)\big)$, where $(T,\mathcal{A}(T),\mu)$ is a complete measurable space and $\big(T,\mathcal{B}(T),\mathcal{M}(T)\big)$, where $\big(T,\mathcal{T}(T)\big)$ is a topological space (see [Frm], 6Bf and 7C).

If $\big(T,\mathcal{A}(T),\mathcal{I}(T)\big)$ is an arbitrary measurable space with negligibles, we can determine its completion $\big(S,\widehat{\mathcal{A}}(T),\mathcal{I}(T)\big)$ by putting:

$$\widehat{\mathcal{A}}(T)=\big\{H\subset T\colon \text{there exist sets } A\in\mathcal{A}(T) \text{ and } B\in\mathcal{A}(T) \text{ such that } A\subset H\subset B \text{ and } B\setminus A\in\mathcal{I}(T)\big\}. \quad (3)$$

Proposition 4. ([Frm, 1D, 1G and 1H]) *Let $\big(T, \mathcal{T}(T), \mathcal{I}(T)\big)$ be a measurable space with negligibles. Then*

(a) $\big(T, \widehat{\mathcal{A}}(T), \mathcal{I}(T)\big)$ *is a complete measurable space with negligibles.*

(b) *If $\big(T, \mathcal{A}(T), \mathcal{I}(T)\big)$ is proto-decomposable, so is $\big(T, \widehat{\mathcal{A}}(T), \mathcal{I}(T)\big)$.*

(c) *If $\big(T, \mathcal{A}(T), \mathcal{I}(T)\big)$ is ω_1-saturated, then it is proto-decomposable.*

(d) *If $\big(T, \mathcal{A}(T), \mathcal{I}(T)\big)$ is complete and proto-decomposable, then $\mathcal{A}(T)$ is closed under the Suslin's operation.*

Let $\big(X, \mathcal{A}(X)\big)$ be a measurable space and let $\big(Y, \mathcal{T}(Y)\big)$ and $\big(Z, \mathcal{T}(Z)\big)$ be topological ones. Let $\mathcal{B}(Y)$ denote the Borel σ-algebra of the space Y. Finally, let $\mathcal{A}(X) \otimes \mathcal{B}(Y)$ stand for a σ-algebra generated by the family of sets $A \times B$, where $A \in \mathcal{A}(X)$ and $B \in \mathcal{B}(Y)$.

Definition 7. A multivalued function $F \colon X \times Y \to Z$ is said to be *product measurable* if it is $\mathcal{A}(X) \otimes \mathcal{B}(Y)$-measurable and $\mathcal{A}(X)$-*sup-measurable* if for each $\mathcal{A}(X)$-measurable multivalued function $G \colon X \to Y$ with closed values the Carathéodory's superposition $\Phi \colon X \to Z$ defined by the formula

$$\Phi(x) = F\big(x, G(x)\big) = \bigcup_{y \in G(x)} F(x, y) \tag{4}$$

is an $\mathcal{A}(X)$-measurable multivalued function.

Let X and Y be two abstract sets. For every set $Z \subset X \times Y$ let us denote by $\operatorname{proj}_X(Z)$ the projection of Z on X, i.e.,

$$\operatorname{proj}_X(Z) = \{x \in X : \text{there exists } y \in Y \text{ such that } (x, y) \in Z\}.$$

For every class $\mathcal{Q}$ of sets let us denote by $\mathbf{A}(\mathcal{Q})$ the class of all sets obtained by applying the operation $\mathbf{A}$ restricted to the class $\mathcal{Q}$.

A class $\mathcal{Q}$ of sets is called compact if for every sequence $(Q_n)_{n \in \mathbb{N}}$ of sets of the family $\mathcal{Q}$, we have $\bigcap_{n \in \mathbb{N}} Q_n \neq \emptyset$ whenever $\bigcap_{i=1}^{n} Q_i \neq \emptyset$ for each $n \in \mathbb{N}$.

Lemma 1. ([M-N, Theorem, p. 161]) *Let X and Y be two abstract sets. Let $\mathcal{X}$ denote a class of subsets of X and $\mathcal{Y}$ a class of subsets of Y.*

If $\mathcal{Y}$ is compact, then $\operatorname{proj}_X(S) \in \mathbf{A}(\mathcal{X})$ for every set $S \in \mathbf{A}(\mathcal{X} \times \mathcal{Y})$.

Lemma 2. *Let $\big(X, \mathcal{A}(X)\big)$ be a measurable space and let (Y, d) be a compact metric one. Then the σ-algebra $\mathcal{A}(X) \otimes \mathcal{B}(Y)$ is contained in the class $\mathbf{A}\big(\mathcal{A}(X) \times \mathcal{K}\big)$, where $\mathcal{K}$ denotes the family of compact subsets of Y.*

This fact is established in Lemma 2.a of [Bie] for the case where Y is the real line. Then it was generalized by D. A. Freedman and J. Neveu for the present case of a compact space Y by substitution of the family $\mathcal{K}$ for the family of compact intervals in Bierlein's proof.

Proposition 5. *Let* $(X, \mathcal{A}(X), \mathcal{I}(X))$ *be a measurable space with negligibles and let* Y *be a Polish one. Then for every set* $M \in \widehat{\mathcal{A}}(X) \otimes \mathcal{B}(Y)$, *it holds* $\operatorname{proj}_X M \in \mathbf{A}(\widehat{\mathcal{A}}(X))$.

P r o o f. Let I denote the Hilbert cube. It is well known (see, e.g., [Kur, p. 249]) that Y is homeomorphic to a subspace $\widehat{Y}$ of I. Then $\mathcal{B}(Y)$ is isomorphic to $\mathcal{B}(\widehat{Y})$, and since $\widehat{Y}$ is G_δ in I (see [Kur, p. 441]), $\widehat{Y} \in \mathcal{B}(I)$ and hence $\mathcal{B}(\widehat{Y}) \subset \mathcal{B}(I)$. Now let $M \in \widehat{\mathcal{A}}(X) \otimes \mathcal{B}(Y)$. Then, using an obvious notation, $\widehat{M} \in \widehat{\mathcal{A}}(X) \otimes \mathcal{B}(\widehat{Y}) \subset \widehat{\mathcal{A}}(X) \otimes \mathcal{B}(I)$. Finally, since I is compact, by Lemma 2 we get $M \in \mathbf{A}(\widehat{\mathcal{A}}(X) \times \mathcal{K})$. Now $\mathcal{K}$ is a compact class on I and, so, by Lemma 1 $\operatorname{proj}_X M = \operatorname{proj}_X \widehat{M} \in \mathbf{A}(\widehat{\mathcal{A}}(X))$ which finishes the proof of Proposition 5. □

Let $(X, \mathcal{A}(X), \mathcal{I}(X))$ be a measurable space with negligibles and let $\widehat{\mathcal{A}}(X)$ be the $\mathcal{I}(X)$-completion of $\mathcal{A}(X)$. We define the following projection property of measurable space with negligibles:

Definition 8. We say that $(X, \mathcal{A}(X), \mathcal{I}(X))$ has the projection property if

$$\operatorname{proj}_X(A) \in \widehat{\mathcal{A}}(X) \text{ whenever } Y \text{ is a Polish space and } A \in \mathcal{A}(X) \otimes \mathcal{B}(Y). \quad (5)$$

Among examples of spaces having projection property (5) there are measure spaces $(X, \mathcal{A}(X), \mu)$ with σ-finite measure μ by the known projection theorem (see, e.g., [C-V, Theorem III. 23]). Then $\widehat{\mathcal{A}}(X)$ is the Lebesgue extension of $\mathcal{A}(X)$.

The spaces $(X, \mathcal{B}(X), \mathcal{M}(X))$, where $(X, \mathcal{T}(X))$ is a topological space, are proto-decomposable, so, by Proposition 4(b) and (d) $\widehat{\mathcal{B}}(X)$ are closed under Suslin's operation, which in turn insures the projection property of $(X, \mathcal{B}(X), \mathcal{M}(X))$ by Proposition 5.

Analogously by Proposition 4(c) each abstract measurable space with negligibles $(X, \mathcal{A}(X), \mathcal{I}(X))$ which is also ω_1-saturated has the projection property (5).

Let $F \colon X \times Y \to Z$ be a multivalued function. Then the multivalued functions $F_x \colon Y \to Z$ defined by $y \to F(x, y)$ and $F^y \colon X \to Z$ defined by $x \to F(x, y)$ are called x-section of F and y-section of F, respectively.

Definition 9. Let $(X, \mathcal{A}(X))$ be a measurable space, $(Y, \mathcal{T}(Y))$ and $(Z, \mathcal{T}(Z))$ topological ones. A multivalued function $F \colon X \times Y \to Z$ is called *lower* (resp. *upper*) *semi-Carathéodory* if for each fixed $y \in Y$ its y-section F^y is $\mathcal{A}(X)$-measurable and for each fixed $x \in X$ its x-section F_x is lower (resp. upper) semicotinuous.

A multivalued function $F \colon X \times Y \to Z$ is said to be *Carathéodory* if it is simultaneously lower and upper semi-Carathéodory.

3. Product-measurability

Theorem 1. *Let $(X, \mathcal{A}(X))$ be a measurable space, Y a Polish space and (Z, ρ) a separable metric one. Suppose that a lower semi-Carathéodory multivalued function $F\colon X \times Y \to Z$ with compact values has all x-sections upper quasicontinuous. Then F is $\mathcal{A}(X) \otimes \mathcal{B}(Y)$-measurable.*

P r o o f . Let $z \in Z$ be fixed. It is sufficient to prove that the real function

$$g_z(x, y) = \rho\bigl(z, F(x, y)\bigr) \quad \text{is} \quad \mathcal{A}(X) \otimes \mathcal{B}(Y)\text{-measurable}. \tag{6}$$

Then by Proposition 3(i) F is weakly $\mathcal{A}(X) \otimes \mathcal{B}(Y)$-measurable and hence by Proposition 2 (ii) it is also $\mathcal{A}(X) \otimes \mathcal{B}(Y)$-measurable.

Let $K(z, r) \subset Z$ be the open ball centered in z with radius $r > 0$ and let $(x, y) \in X \times Y$ be a fixed point. According to the assumption, F^y is $\mathcal{A}(X)$-measurable, so, by Proposition 2 (i) F^y is weakly $\mathcal{A}(X)$-measurable. Therefore $(F^y)^-\bigl(K(z, r)\bigr) \in \mathcal{A}(X)$. Furthermore,

$$\begin{aligned} (F^y)^-\bigl(K(z, r)\bigr) &= \bigl\{x \in X : F^y(x) \cap K(z, r) \neq \emptyset\bigr\} \\ &= \bigl\{x \in X : \rho\bigl(z, F^y(x)\bigr) < r\bigr\} = \bigl(g_z^y\bigr)^{-1}(-\infty, r). \end{aligned}$$

Therefore we have

$$\bigl(g_z^y\bigr)^{-1}(-\infty, r) \in \mathcal{A}(X) \quad \text{for each } y \in Y, \text{ i.e., all } y\text{-sections of } g_z \text{ are } \mathcal{A}(X)\text{-measurable}. \tag{7}$$

Moreover, multivalued function F_x is lower semicontinuous. Thus $(F_x)^-\bigl(K(z, r)\bigr)$ is an open subset of Y. Since

$$\begin{aligned} \bigl((g_z)_x\bigr)^{-1}(-\infty, r) &= \bigl\{y \in Y : \rho\bigl(z, F_x(y)\bigr) < r\bigr\} \\ &= \bigl\{y \in Y : F_x(y) \cap K(z, r) \neq \emptyset\bigr\} = (F_x)^-\bigl(K(z, r)\bigr) \end{aligned}$$

we conclude that $\bigl((g_z)_x\bigr)^{-1}(-\infty, r)$ is an open subset of Y, hence

$$\bigl((g_z)_x\bigr)^{-1}(-\infty, r) \text{ is an open subset of } Y \text{ for each } x \in X, \text{ i.e., all } x\text{-sections of } g_z, \text{ are upper semicontinuous}. \tag{8}$$

Since F_x is also upper quasicontinuous at y, by Proposition 1, there exists a quasi-open set $A(y)$ containing y such that the restriction $F_x\big|_{A(y)}$ is upper semicontinuous at y. Therefore

$$\text{there exists an open set } V(y) \text{ such that } V(y) \subset A(y) \subset \mathrm{Cl}\bigl(V(y)\bigr), \; y \in \mathrm{Cl}\bigl(V(y)\bigr), \text{ and } F_x\big|_{V(y) \cup \{y\}} \text{ is continuous at } y. \tag{9}$$

Let $S = \{s_1, s_2, \dots\}$ be a dense subset of Y. Then by (9), to each point $(x,y) \in X \times Y$ there corresponds a sequence $\big(s_n(x,y)\big)_{n\in\mathbb{N}}$ such that

$$s_n(x,y) \in S, \quad \lim_{n\to\infty} s_n(x,y) = y$$

and

$$\lim_{n\to\infty} \rho\big(z, F(x, s_n(x,y))\big) = \rho\big(z, F(x,y)\big)\,,$$

i.e.,

$$\lim_{n\to\infty} g_z\big(x, s_n(x,y)\big) = g_z(x,y)\,. \tag{10}$$

Now we define a multifunction $G\colon X \to Y \times \mathbb{R}$ as follows:

$$G(x) = \big\{(y,r) \in Y \times \mathbb{R} \colon g_z(x,y) \geq r\big\}\,.$$

By virtue of (8)

$$\text{the values of the multivalued function } G \text{ are closed subsets of } Y \times \mathbb{R}\,. \tag{11}$$

We will show that

$$\text{the multifunction } G \text{ is weakly } \mathcal{A}(X)\text{-measurable}. \tag{12}$$

Let $(q_m)_{m\in\mathbb{N}}$ be an enumeration of the rational numbers. Define a sequence of functions $f_{n,m}\colon X \to Y \times \mathbb{R}$ by the formula

$$f_{n,m}(x) = \Big(s_n(x,y),\, \min\big(q_m, g_z\big(x, s_n(x,y)\big)\big)\Big)\,.$$

It is clear that

$$\begin{gathered}\text{all functions } f_{n,m}\colon X \to Y \times \mathbb{R} \text{ are } \mathcal{A}(X)\text{-measurable and} \\ f_{n,m}(x) \in G(x) \text{ for each } x \in X \text{ and } n, m \in \mathbb{N}\,.\end{gathered} \tag{13}$$

Thus

$$\big\{f_{n,m}(x) \colon n,m \in \mathbb{N}\big\} \subset G(x) \quad \text{for each } x \in X\,. \tag{14}$$

Since $G(x)$ is closed (see (11)), we have

$$\mathrm{Cl}\big(\big\{f_{n,m}(x) \colon n,m \in \mathbb{N}\big\}\big) \subset G(x)\,. \tag{15}$$

Now let $(y,r) \in G(x)$, i.e., $g_z(x,y) \geq r$. We can choose the sequence $(q_m)_{m\in\mathbb{N}}$ such that $q_m \leq g_z\big(x, s_n(x,y)\big)$ for each $m \in \mathbb{N}$, and $\lim\limits_{m\to\infty} q_m = r$. Then $\lim\limits_{n\to\infty} \lim\limits_{m\to\infty} f_{n,m}(x) = (y,r)$, this means $(y,r) \in \mathrm{Cl}\big(\big\{f_{n,m}(x) \colon n,m \in \mathbb{N}\big\}\big)$ and $G(x) \subset \mathrm{Cl}\big(\big\{f_{n,m}(x) \colon n,m \in \mathbb{N}\big\}\big)$. Thus by (15)

$$G(x) = \mathrm{Cl}\big(\big\{f_{n,m}(x) \colon n,m \in \mathbb{N}\big\}\big)\,. \tag{16}$$

Now (12) is a simple consequence of (13), (16), and Proposition 3 (ii). The statement (12) has been proved.

Then by Proposition 3 (i),

$$\operatorname{Gr}(G) = \big\{(x,y,r) \in X \times Y \times \mathbb{R} \colon (y,r) \in G(x)\big\} \in \mathcal{A}(X) \otimes \mathcal{B}(Y \times \mathbb{R}) .$$

and then

$$\big(\operatorname{Gr}(G)\big)^r = \big\{(x,y) \in X \times Y \colon (x,y,r) \in \operatorname{Gr}(G)\big\} \in \mathcal{A}(X) \otimes \mathcal{B}(Y) . \tag{17}$$

Let us note that

$$\begin{aligned}\big(\operatorname{Gr}(G)\big)^r &= \big\{(x,y) \in X \times Y \colon (y,r) \in G(x)\big\}\\ &= \big\{(x,y) \in X \times Y \colon g_z(x,y) \ge r\big\}\\ &= X \times Y \setminus \big\{(x,y) \in X \times Y \colon g_z(x,y) < r\big\}\\ &= X \times Y \setminus g_z^{-1}(-\infty, r) .\end{aligned}$$

Therefore by (17) $g_z^{-1}(-\infty,r) \in \mathcal{A}(X) \otimes \mathcal{B}(Y)$, so, (6) is shown and the proof of Theorem 2 is finished. ☐

4. Sup-measurability

THEOREM 2. *Let $\big(X, \mathcal{A}(X), \mathcal{I}(X)\big)$ be a measurable space with negligibles which has the projection property (5). Let Y be a Polish space and $\big(Z, \mathcal{T}(Z)\big)$ a topological one. Then every $\mathcal{A}(X) \otimes \mathcal{B}(Y)$-measurable multivalued function $F \colon X \times Y \to Z$ is $\widehat{\mathcal{A}}(X)$-sup-measurable.*

P r o o f . Let $H \colon X \to Y$ be an $\widehat{\mathcal{A}}(X)$-measurable multivalued function with closed values. We will show that the multivalued function $\Phi(x) = F\big(x, H(x)\big)$ is $\widehat{\mathcal{A}}(X)$-measurable. Let $D \subset Z$ be a closed set. By $\mathcal{A}(X) \otimes \mathcal{B}(Y)$-measurability of F,

$$F^-(D) = \big\{(x,y) \in X \times Y \colon F(x,y) \cap D \ne \emptyset\big\} \in \mathcal{A}(X) \otimes \mathcal{B}(Y) . \tag{18}$$

On the other hand, in view of $\widehat{\mathcal{A}}(X)$-measurability of H and by Proposition 2 (i), H is weakly $\widehat{\mathcal{A}}(X)$-measurable, and by Proposition 3 (i),

$$\operatorname{Gr}(H) = \big\{(x,y) \in X \times Y \colon y \in H(x)\big\} \in \widehat{\mathcal{A}}(X) \otimes \mathcal{B}(Y) . \tag{19}$$

Let us remark that

$$\big\{(x,y) \in X \times Y \colon F(x,y) \cap D \ne \emptyset \text{ and } y \in H(x)\big\} = F^-(D) \cap \operatorname{Gr}(H) .$$

Thus by (18) and (19)

$$\{(x,y) \in X \times Y \colon F(x,y) \cap D \neq \emptyset \text{ and } y \in H(x)\} \in \widehat{\mathcal{A}}(X) \otimes \mathcal{B}(Y).$$

Hence, using (5) we arrive at the following conclusion:

$$\begin{aligned}\Phi^-(D) &= \{x \in X \colon F(x, H(x)) \cap D \neq \emptyset\} \\ &= \{x \in X \colon F(x,y) \cap D \neq \emptyset \text{ and } y \in H(x)\} \\ &= \operatorname{proj}_X \{(x,y) \in X \times Y \colon F(x,y) \cap D \neq \emptyset \text{ and } y \in H(x)\} \in \widehat{\mathcal{A}}(X)\end{aligned}$$

which finishes the proof of Theorem 2. □

The following Corollary is an immediate conclusion of Theorem 1 and Theorem 2.

COROLLARY 1. *Let $(X, \mathcal{A}(X), \mathcal{I}(X))$ be a measurable space with negligibles which has the projection property (5). Let Y be a Polish space and (Z, ρ) a separable metric one. If $F \colon X \times Y \to Z$ is a lower semi-Carathéodory multivalued function with compact values whose every x-section is upper quasicontinuous, then F is $\widehat{\mathcal{A}}(X)$-sup-measurable.*

Remark 1. If a multivalued function is only lower semi-Carathéodory, then it may not be sup-measurable.

Consider, for example, a multifunction $F \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as follows:

$$F(x,y) = \begin{cases} [-1,2] & \text{if } x \neq y, \\ [-1,0] & \text{if } x = y \text{ and } x \in A, \\ [1,2] & \text{if } x = y \text{ and } x \in \mathbb{R} \setminus A, \end{cases}$$

where $A \notin \mathcal{L}(\mathbb{R})$. It is clear that the above multifunction fulfils the mentioned conditions. But taking measurable multivalued function $G(x) = \{x\}$, the superposition $\Phi(x) = F(x, G(x)) = F(x,x)$ is not Lebesgue measurable.

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Received November 29, 2002

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