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ON PT-SPACES

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ABSTRACT. The research in the selection theory led to define some class of topological spaces, so called pt-spaces. In this paper the relations between pt-spaces and other classes of ones are studied. It is also shown that each Hausdorff pt-space is a Baire space and that any locally compact paracompact space is a pt-space.

Throughout the whole paper $(X,\mathcal{T})$ or simply X denotes a topological space on which no separation axioms are assumed unless it is explicitly stated. Thus, similarly as in [1], compact, locally compact and paracompact spaces need not be T_2.

A subset A of a topological space X is said to be:

- *regularly open* (*regularly closed*) if $A = \mathrm{Int}(\overline{A})$, (resp. $A = \overline{\mathrm{Int}(A)}$); sometimes the names "open domain" and "closed domain" are also used [5, 3];
- *semi-open*, if there exists an open set U satisfying $U \subset A \subset \overline{U}$ (equivalently $A \subset \overline{\mathrm{Int}(A)}$), [6];
- *regularly semi-open* if there exists a regularly open set U such that $U \subset A \subset \overline{U}$, [4].

The relations among these classes of sets are illustrated by the following diagram

$$\begin{array}{ccc} \text{regularly open} & \longrightarrow & \text{open} \\ \downarrow & & \downarrow \\ \text{regularly semi-open} & \longrightarrow & \text{semi-open} \end{array}$$

The simple examples show that all inclusions in the above diagram are proper. In the sequel, some properties of these sets will be used; we list them as a lemma.

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Lemma 1.

(a) *The complement of a regularly open (regularly closed) set is regularly closed (regularly open),* [5].

(b) *The interior (closure) of a closed (resp. open) set is regularly open (resp. regularly closed),* [5].

(c) *The closure of any semi-open set is regularly closed,* [4].

(d) *If A is regularly semi-open and B is regularly open, then $A \cap B$ is regularly semi-open* [4].

(e) *The union of a locally finite family of regularly closed sets is regularly closed.*

A topological space X is called a *pt-space* if there exists a cover $\{W_j\colon j \in J\}$ of X consisting of pairwise disjoint regularly semi-open sets with $\overline{W_j}$ compact for $j \in J$, [4].

Theorem 1. *Every Hausdorff pt-space is a Baire space.*

P r o o f. Let X be a Hausdorff pt-space and let $\mathcal{A}$ be a cover of X consisting of pairwise disjoint regularly semi-open sets with compact closure. By $\{W_n\colon n \in \mathbb{N}\}$ denote a sequence of open dense sets in X. For any open nonempty set $U \subset X$ and a point $x \in U$ there is $A \in \mathcal{A}$ such that $x \in A$. Hence $V = U \cap \operatorname{Int}(A) \neq \emptyset$, $\overline{V}$ is compact and $V \cap W_n \neq \emptyset$ for $n \in \mathbb{N}$. Let $\widetilde{V \cap W_n}$ be the closure of $V \cap W_n$ in the subspace $\overline{V}$. Then $\widetilde{V \cap W_n} = \overline{V \cap W_n} \cap \overline{V} = \overline{V}$, so $V \cap W_n$ is an open dense subset of the compact Hausdorff space $\overline{V}$. Hence $\bigcap_{n=1}^{\infty} V \cap W_n$ is dense in $\overline{V}$. Thus we obtain $\emptyset \neq V \cap \bigcap_{n=1}^{\infty} W_n \subset U \cap \bigcap_{n=1}^{\infty} W_n$, i.e., $\bigcap_{n=1}^{\infty} W_n$ is dense in X which completes the proof. □

Theorem 2. *A separable pt-space does not contain a closed discrete subspace of the cardinality continuum.*

P r o o f. On the contrary suppose that a separable pt-space X contains a closed discrete subspace L of the cardinality continuum. Since L is discrete, for each compact set $M \subset X$ the set $M \cap L$ is at most finite. Let $\mathfrak{A}$ be a regularly semi-open cover of X consisting of pairwise disjoint sets such that $\overline{U}$ is compact for $U \in \mathfrak{A}$. For each $p \in L$ we fix $U_p \in \mathfrak{A}$ with $p \in U_p$. The set $\overline{U_p}$ is compact, so $U_p \cap L$ is nonempty finite for $p \in L$ and $L = \bigcup_{p \in L} U_p \cap L$. From this it follows that the cardinality of $\mathfrak{A}$ is at least continuum. Let D be a countable dense subset of X. For each $U \in \mathfrak{A}$ we have $D \cap U \neq \emptyset$, so $\{U \cap D\colon U \in \mathfrak{A}\}$ is a family of pairwise disjoint nonempty sets. Furthermore $D = D \cap \bigcup_{U \in \mathfrak{A}} U = \bigcup_{U \in \mathfrak{A}} D \cap U$ which is a contradiction finishing the proof. □

As an immediate consequence of Theorem 2 we obtain:

EXAMPLES 1.

(a) The Niemytzki plane (comp. [3, Examples 1.2.4, 1.3.9, 1.4.5, 1.5.9]) is not a pt-space.

(b) Let X be the Sorgenfrey line; then $X \times X$ is not a pt-space.

Also, various other well-known spaces do not have the considered property.

EXAMPLES 2.

(a) Every infinite dimensional Hausdorff topological vector space is not a pt-space.

For instance, in such space a neighbourhood V of zero with $\overline{V}$ compact does not exist [2]. In the consequence, for each subset A such that $\operatorname{Int}(A) \neq \emptyset$ the closure of A is not compact.

(b) $\mathbb{R}^n$ with the ordinary density topology τ_d is not a pt-space. In the space $(\mathbb{R}^n, \tau_d)$ each compact set is of the Lebesgue measure zero, while the measure of an open nonempty set is positive [7].

From above examples it follows that Theorem 1 is not invertible.

THEOREM 3. *Every locally compact second countable space is a pt-space.*

Proof. Let $\mathcal{B}$ be a countable base of a locally compact space X. For each $x \in X$ we choose its neighbourhood belonging to $\mathcal{B}$ and possessing the compact closure. Thus we obtain an open cover $\mathfrak{A} = \{U_n : n \in \mathbb{N}\}$ of X with $\overline{U}_n$ compact for $n \in \mathbb{N}$. Now let

$$W_1 = \overline{U}_1, \quad W_2 = \overline{U}_2 \backslash \overline{U}_1, \quad \dots, \quad W_n = \overline{U}_n \setminus \bigcup_{k=1}^{n-1} \overline{U}_k, \quad \dots$$

Then $\{W_n : n \in \mathbb{N}\}$ is a regularly semi-open cover of X, $W_n \cap W_m = \emptyset$ for $n, m \in \mathbb{N}$, $n \neq m$ and $\overline{W}_n$ is compact for $n \in \mathbb{N}$. $\square$

In [4] I. Kupka stated the following question: is every regular locally compact topological space a pt-space?

In the next theorem some connections between pt-spaces and the local compactness are formulated. At first we give the necessary definition.

A topological space X is called almost paracompact if for each open cover $\mathfrak{A}$ of X there is a locally finite family $\mathcal{V}$ of open sets which refines $\mathfrak{A}$ and whose union is dense in X, [9].

THEOREM 4. *In a topological space X the following conditions are equivalent:*

(a) *there exists a locally finite cover with properties from the definition of a pt-space;*

(b) *there exists a locally finite cover consisting of regularly closed compact sets;*

(c) *there exists a locally finite cover consisting of closed compact sets;*

(d) *there exists a locally finite open cover consisting of the sets which closure is compact;*

(e) *X is a locally compact and paracompact space;*

(f) *X is a locally compact and almost paracompact space.*

Proof. We will prove that (a) $\Rightarrow$ (b). Let $\{M_j : j \in J\}$ be a locally finite cover of X consisting of pairwise disjoint regularly semi-open sets such that $\overline{M}_j$ is compact for $j \in J$. Then the family $\{\overline{M}_j : j \in J\}$ has all properties mentioned in (b).

The implication (b) $\Rightarrow$ (c) is obvious.

Now we will show that (c) $\Rightarrow$ (d). Let $\mathcal{F} = \{F_j : j \in J\}$ be a locally finite closed compact cover of X. For each point $x \in X$ there is a neighbourhood U intersecting only finite number of sets $F_{j,1}, \dots, F_{j,n} \in \mathcal{F}$. Then $\overline{U} \subset F_{j,1} \cup \dots \cup F_{j,n}$, so $\overline{U}$ is compact. Thus we have shown that X is locally compact.

For each $j \in J$ and $x \in F_j$ there is a neighbourhood $W_{j,x}$ which meets only finite number of sets belonging to $\mathcal{F}$. Since F_j are compact we can select a finite sequence of points $x_{j,1}, \dots, x_{j,n(j)} \in F_j$ such that $F_j \subset W_{j,x_{j,1}} \cup \dots \cup W_{j,x_{j,n(j)}}$. This means that $W_{j,x_{j,1}} \cup \ldots \cup W_{j,x_{j,n(j)}}$ is a neighbourhood of F_j which intersects only finite number of sets from $\mathcal{F}$. Hence for each $j \in J$ the set $A_j = \{i \in J : F_j \cap F_i \neq \emptyset\}$ is nonempty finite. The family $\mathcal{F}$ is locally finite, therefore for each $z \in J$ the set $\bigcup_{j \notin A_z} F_j$ is closed and disjoint with F_z. Since X is locally compact an open set U_z can be chosen such that $F_z \subset U_z$, $\overline{U}_z$ is compact and $U_z \cap \bigcup_{j \notin A_z} F_j = \emptyset$. This implies that for every two indices $j_1, j_2 \in J$ for which $U_{j_1} \cap F_{j_2} \neq \emptyset$ it holds $F_{j_1} \cap F_{j_2} \neq \emptyset$. Hence $\{j \in J : U_j \cap F_z \neq \emptyset\} = A_z$. The family $\mathfrak{A} = \{U_j : j \in J\}$ is an open cover of X and $\overline{U}_j$ is compact for $j \in J$. It remains to show that $\mathfrak{A}$ is locally finite. Let us take any $z \in J$. If $U_j \cap U_z \neq \emptyset$ for certain $j \in J$, then there exists $F_i \in \mathcal{F}$ such that $U_j \cap U_z \cap F_i \neq \emptyset$. Thus $F_j \cap F_i \neq \emptyset \neq F_z \cap F_i$ which gives $i \in A_z$ and $j \in A_i$. This means that $\{j \in J : U_j \cap U_z \neq \emptyset\} \subset \bigcup_{i \in A_z} A_i$. So we have shown that for each $z \in J$ the set $\{j \in J : U_j \cap U_z \neq \emptyset\}$ is finite, i.e., the cover $\mathfrak{A}$ is locally finite.

To prove the implication (d) $\Rightarrow$ (e), we take a locally finite open cover $\mathfrak{A} = \{U_j : j \in J\}$ of X such that $\overline{U}_j$ is compact for $j \in J$. The existence of such cover means that X is locally compact. We will show that X is paracompact. Let $\mathcal{V} = \{V_s : s \in S\}$ be an open cover of X. Since X is locally compact, for each $j \in J$ and for each $x \in \overline{U}_j$ we can choose an open set $W_{j,x}$ such that $x \in W_{j,x}$, $\overline{W}_{j,x}$ is compact and $W_{j,x}$ is contained in some set belonging to $\mathcal{V}$. For each $j \in J$, using the compactness of $\overline{U}_j$, we can select a finite sequence of points $x_{j,1}, \dots, x_{j,n(j)} \in \overline{U}_j$ such that $\overline{U}_j \subset W_{j,x_{j,1}} \cup \dots \cup W_{j,x_{j,n(j)}}$. Let us put $\widehat{W}_{j,x_{j,k}} = W_{j,x_{j,k}} \cap U_j$ for $j \in J$, $k = 1, \dots, n(j)$; then $U_j = \widehat{W}_{j,x_{j,1}} \cup \dots \cup \widehat{W}_{j,x_{j,n(j)}}$. The cover $\mathcal{B} = \left\{\widehat{W}_{j,x_{j,k}} : j \in J,\ k = 1, \dots, n(j)\right\}$ is locally finite because it refines $\mathfrak{A}$. As it follows from the construction of $\mathcal{B}$, it also refines $\mathcal{V}$; thus we have shown that X is a paracompact space.

The implication (e) $\Rightarrow$ (f) is evident since each paracompact space is almost paracompact.

Finally, we will prove that (f) $\Rightarrow$ (a). For each $x \in X$ we fix a neighbourhood U_x of this point such that $\overline{U}_x$ is compact. By the assumption, there is a locally finite open refinement $\mathcal{V}$ of $\mathfrak{A} = \{U_x : x \in X\}$ such that $\overline{\bigcup_{V \in \mathcal{V}} V} = X$. Then $\mathcal{V}_1 = \{\overline{V} : V \in \mathcal{V}\}$ is a locally finite cover of X consisting of regularly closed compact sets. Let "$<$" be a well order in $\mathcal{V}_1$ and let α be the order type of $(\mathcal{V}_1, <)$. Then $\mathcal{V}_1$ can be represented as the transfinite sequence

$$\overline{V}_0, \overline{V}_1, \dots, \overline{V}_\xi, \dots; \qquad \xi < \alpha.$$

Let us put

$$W_0 = \overline{V}_0, \quad W_1 = \overline{V}_1 \setminus \overline{V}_0, \quad \dots, \quad W_\xi = \overline{V}_\xi \Big\backslash \bigcup_{\beta<\xi} \overline{V}_\beta, \quad \dots; \qquad \xi < \alpha.$$

Since $W_\xi \subset \overline{V}_\xi$, each of the sets $\overline{W}_\xi$ is compact. Furthermore, $\{W_\xi : \xi < \alpha\}$ refines $\mathcal{V}_1$, so $\{W_\xi : \xi < \alpha\}$ is locally finite. For any $\xi < \alpha$ the family $\{\overline{V}_\beta : \beta < \xi\}$ is locally finite, thus $\bigcup_{\beta<\xi} \overline{V}_\beta = \overline{\bigcup_{\beta<\xi} V_\beta}$. Hence $\bigcup_{\beta<\xi} \overline{V}_\beta$ is regularly closed. In consequence, W_ξ is the intersection of regularly semi-open set $\overline{V}_\xi$ and regularly open set $X \setminus \bigcup_{\beta<\xi} \overline{V}_\beta$, so W_ξ is regularly semi-open. Now, let us take any point $x \in X$. Since $\mathcal{V}_1$ is a cover of X, $\mathcal{V}_x = \{\overline{V}_\xi : x \in \overline{V}_\xi\}$ is a nonempty subset of the well ordered set $(\mathcal{V}_1, <)$. Therefore there is $\xi < \alpha$ such that $x \in \overline{V}_\xi$ and $x \notin \overline{V}_\beta$ for $\beta < \xi$. It means $x \in W_\xi$, i.e., $\{W_\xi : \xi < \alpha\}$ is a cover of X possessing all properties from the definition of a pt-space. Thus the proof is completed. □

Corollary 1. *Every locally compact metric space is a pt-space.*

On the other hand a metric pt-space need not be locally compact.

Example 3. Let us consider $X = \big([0,1]\times\{0\}\big) \cup \bigcup_{n=2}^{\infty}\Big(\{\frac{1}{n}\}\times\big[\frac{1}{n},1\big]\Big)$ with the euclidean metric. The sets $A_1 = [0,1]\times\{0\}$, $A_n = \{\frac{1}{n}\}\times\big[\frac{1}{n},1\big]$, $n \geq 2$, are regularly closed compact pairwise disjoint and $X = \bigcup_{n=1}^{\infty} A_n$. Thus X is a pt-space. But none neighbourhood of the point $p = (0,0)$ has the compact closure, so X is not locally compact.

Taking the last example into account it seems to be sensible to use the following definition: a topological space X is said to be locally compact at a point $x \in X$ if there is a neighbourhood U of x such that the set $\overline{U}$ is compact. We set

$$E(X) = \{x \in X\colon \ \text{the space } X \text{ is not locally compact at } x\}.$$

Then we immediately obtain:

- $E(X) = \emptyset$ if and only if X is locally compact;
- $E(X)$ is a closed set;
- $E(X) \subset X^d$, where X^d is the derived set of X;
- if X is a pt-space then $E(X)$ is a nowhere dense subset of X.

Now, let (X,d) be a metric space. By $B(x,r)$ we denote the open ball with center x and radius r. For two sets $A_1, A_2 \subset X$ we write $\operatorname{dist}(A_1,A_2) = \inf_{x\in A_1, y\in A_2} d(x,y)$.

Theorem 5. *Let (X,d) be a dense in itself metric space. If the set $E(X)$ is discrete, then X is a pt-space.*

Proof. Under assumptions, $\big\{\{x\}\colon x \in E(X)\big\}$ is a discrete family of closed sets. By the paracompactness of X, there exists a discrete family $\big\{U_x\colon x \in E(X)\big\}$ of pairwise disjoint open sets with $x \in U_x$ for $x \in E(X)$, [3, Th. 5.1.18 and 5.1.17]. For each $x \in E(X)$ we fix $r(x) > 0$ such that $\overline{B}\big(x, r(x)\big) \subset U_x$. Thus we get the discrete family of pairwise disjoint sets $\big\{\overline{B}\big(x, r(x)\big)\colon x \in E(X)\big\}$. Since $E(X) \subset X^d$, for each $x \in E(X)$ we choose a sequence $\big\{y_n(x)\colon n \in \mathbb{N}\big\}$ of points belonging to $B\big(x, r(x)\big)$ and a sequence $\big\{r_n(x)\colon x \in \mathbb{N}\big\}$ of numbers satisfying the following conditions:

$$0 < r_n(x) < \min\big\{r(x), d\big(x, y_n(x)\big)\big\},\ n \in \mathbb{N},$$

$$d\big(x, y_n(x)\big) \searrow 0 \text{ and } r_n(x) \searrow 0, \text{ whenever } n \to \infty,$$

$$\overline{B}\big(y_n(x), r_n(x)\big) \subset B\big(x, r(x)\big),\ n \in \mathbb{N},$$

$$\overline{B}\big(y_n(x), r_n(x)\big) \text{ is compact},\ n \in \mathbb{N},$$

$$\operatorname{dist}\Big(B\big(y_n(x), r_n(x)\big),\ B\big(y_m(x), r_m(x)\big)\Big) > 0 \text{ for } n, m \in \mathbb{N},\ n \neq m.$$

Now, we put

$$A(x) = \{x\} \cup \bigcup_{n=1}^{\infty} \overline{B}\big(y_n(x), r_n(x)\big) \text{ for } x \in E(X),$$
$$\text{and} \quad \mathcal{V}_1 = \big\{A(x) \colon x \in E(X)\big\}.$$

Then $\mathcal{V}_1$ is a discrete family consisting of pairwise disjoint regularly closed compact sets, so $\bigcup_{x \in E(X)} A(x)$ is regularly closed. Let $Z = X \setminus \bigcup_{x \in E(X)} A(x)$. For each $z \in Z$ we choose a neighbourhood W_z such that $\overline{W}_z$ is compact and $\overline{W}_z \subset Z$. Thus $\mathcal{W} = \{W_z \colon z \in Z\}$ is an open cover of Z. Since Z is an open paracompact subspace of X, there exists an open in X and locally finite in Z cover $\mathcal{V}_2$ of Z which refines $\mathcal{W}$. Hence for each $V \in \mathcal{V}_2$ the set $\overline{V}$ is compact and $\overline{V} \subset Z$. Let $\widetilde{M}$ denote the closure of a set $M \subset Z$ in the subspace Z. Then for each set $V \in \mathcal{V}_2$ we have $\widetilde{V} = \overline{V} \cap Z = \overline{V}$. Now let "$<$" be a well order in $\mathcal{V} = \{\overline{V} \colon V \in \mathcal{V}_1 \cup \mathcal{V}_2\}$ and let α be the order type of $(\mathcal{V}, <)$. Then $\mathcal{V}$ can be represented as the transfinite sequence

$$\overline{V}_0, \overline{V}_1, \dots, \overline{V}_\xi, \dots; \qquad \xi < \alpha.$$

We put:

$$G_0 = \overline{V}_0, \quad G_1 = \overline{V}_1 \setminus \overline{V}_0, \quad \dots, \quad G_\xi = \overline{V}_\xi \Big\backslash \bigcup_{\beta < \xi} \overline{V}_\xi, \quad \dots; \qquad \xi < \alpha.$$

The sets $\overline{G}_\xi$ are compact, because $G_\xi \subset \overline{V}_\xi$ for $\xi < \alpha$. Now,we show that the sets G_ξ are regularly semi-open in X. For any $V_\beta \in \mathcal{V}_1$ we have $\overline{V}_\beta = V_\beta$, hence

$$\bigcup_{\beta < \xi} \overline{V}_\beta = \bigcup_{\beta < \xi} \{V_\beta \colon V_\beta \in \mathcal{V}_1\} \cup \bigcup_{\beta < \xi} \{\overline{V}_\beta \colon V_\beta \in \mathcal{V}_2\}.$$

Since $\{V_\beta \colon V_\beta \in \mathcal{V}_1, \beta < \xi\}$ is a discrete family of regularly closed sets, the set $D_1 = \bigcup_{\beta < \xi} \{V_\beta \colon V_\beta \in \mathcal{V}_1\}$ is regularly closed in X. For any $V_\beta \in \mathcal{V}_2$ we have $\overline{V}_\beta = \widetilde{V}_\beta$; furthermore, $\mathcal{V}_2$ is locally finite in Z, so

$$\bigcup_{\beta < \xi} \{\overline{V}_\beta \colon V_\beta \in \mathcal{V}_2\} = \bigcup_{\beta < \xi} \{\widetilde{V}_\beta \colon V_\beta \in \mathcal{V}_2\}$$
$$= \widetilde{\bigcup_{\beta < \xi} \{V_\beta \colon V_\beta \in \mathcal{V}_2\}} = Z \cap \overline{\bigcup_{\beta < \xi} \{V_\beta \colon V_\beta \in \mathcal{V}_2\}}.$$

The set $D_2 = \overline{\bigcup_{\beta < \xi} \{V_\beta \colon V_\beta \in \mathcal{V}_2\}}$ is regularly closed in X and we obtain $\bigcup_{\beta < \xi} \overline{V}_\beta = D_1 \cup (Z \cap D_2)$. Hence

$$G_\xi = \overline{V}_\xi \setminus \big(D_1 \cup (Z \cap D_2)\big) = \big(\overline{V}_\xi \setminus D_1\big) \cap \big(\overline{V}_\xi \setminus Z \cap D_2\big).$$

If $V_\xi \in \mathcal{V}_1$, then $V_\xi \cap Z = \emptyset$; for $V_\xi \in \mathcal{V}_2$ we have $\overline{V}_\xi \subset Z$. Thus

$$\overline{V}_\xi \setminus (Z \cap D_2) = \big(\overline{V}_\xi \setminus Z\big) \cup \big(\overline{V}_\xi \setminus D_2\big) = \begin{cases} \overline{V}_\xi \cup \big(\overline{V}_\xi \setminus D_2\big) = \overline{V}_\xi & \text{if } V_\xi \in \mathcal{V}_1\,, \\ \overline{V}_\xi \setminus D_2 & \text{if } V_\xi \in \mathcal{V}_2\,. \end{cases}$$

In consequence, we have

$$G_\xi = \begin{cases} (\overline{V}_\xi \setminus D_1) \cap \overline{V}_\xi & \text{if } V_\xi \in \mathcal{V}_1\,, \\ (\overline{V}_\xi \setminus D_1) \cap (\overline{V}_\xi \setminus D_2) & \text{if } V_\xi \in \mathcal{V}_2\,, \end{cases}$$

i.e.,

$$G_\xi = \begin{cases} (\overline{V}_\xi \setminus D_1) & \text{if } V_\xi \in \mathcal{V}_1\,, \\ \overline{V}_\xi \setminus (D_1 \cup D_2) & \text{if } V_\xi \in \mathcal{V}_2\,. \end{cases}$$

The sets $\overline{V}_\xi$, D_1, $D_1 \cup D_2$ are regularly closed, so G_ξ is regularly semi-open. To complete the proof it remains to show that $\mathcal{G} = \{G_\xi \colon \xi < \alpha\}$ is a cover of X. Let $x \in X$. Since $\mathcal{V}$ is a cover of X we have $\{\overline{V}_\xi \in \mathcal{V} \colon x \in \overline{V}_\xi\} \neq \emptyset$. Hence there is $\xi < \alpha$ such that $x \in \overline{V}_\xi$ and $x \notin \overline{V}_\beta$ for $\beta < \xi$. This gives $x \in G_\xi$. Thus we have shown that $\mathcal{G}$ has all properties from the definition of a pt-space which completes the proof. □

Now, let us consider the following:

EXAMPLE 4. Let $(X, \mathcal{T})$ be a T_1 topological space and let $Y = X \times [0,1]$. For each point $p = (x,y) \in Y$ with $y > 0$ and for each $\varepsilon > 0$ we put

$$W_\varepsilon(p) = \big\{(x,r) \in Y \colon 0 < r \leq 1,\ |y - r| < \varepsilon\big\}\,,$$

and

$$\mathcal{B}(p) = \big\{W_\varepsilon(p) \colon \varepsilon > 0\big\}\,.$$

If $p = (x,0) \in Y$, then for $\varepsilon > 0$ and for a $\mathcal{T}$-neighbourhood U of x we set

$$V_{U,\varepsilon}(p) = \big\{(z,r) \in Y \colon z \in U,\ 0 \leq r < \varepsilon\big\}$$

and

$$\mathcal{B}(p) = \big\{V_{U,\varepsilon}(p) \colon \varepsilon > 0\,,\ U \text{ is a } \mathcal{T}\text{-neighbourhood of } p\big\}\,.$$

The collection $\big\{\mathcal{B}(p) \colon p \in Y\big\}$ is a neighbourhood system determining some topology τ on Y. Then we have

(I) $(X, \mathcal{T})$ is homeomorphic to $\big(X \times \{0\},\, \tau_{|X\times\{0\}}\big)$ and $X \times \{0\}$ is a closed subspace of Y.

Let $x \in X$. The set $A_x = \{x\} \times (0,1]$ is open in Y, $\overline{A}_x = \{x\} \times [0,1]$ is compact regularly semi-open and $Y = \bigcup_{x\in X} \overline{A}_x$. Evidently $\overline{A}_{x_1} \cap \overline{A}_{x_2} = \emptyset$ if $x_1 \neq x_2$. So we have shown:

(II) (Y, τ) is a pt-space.

For each neighbourhood $V_{U,\varepsilon}(p)$ of any point $p \in X \times \{0\}$ it holds: $\overline{V}_{U,\varepsilon}(p) = \big(U \times [0,\varepsilon]\big) \cup \big(\overline{U} \times \{0\}\big)$. Thus we get:

(III) If $(X,\mathcal{T})$ is a dense in itself T_1 space, then $E(Y) = X \times \{0\}$.

(IV) Let $(X,\mathcal{T})$ be a dense in itself T_1 space which is neither locally compact nor paracompact (for example the square of the Sorgenfrey line). Then the pt-space (Y,τ) is neither locally compact nor paracompact.

COROLLARIES 2.

(a) *The property "to be a pt-space" is not hereditary with respect to closed subspaces.*

(b) *Any topological property P hereditary with respect to closed subsets is not a necessary condition for to be a pt-space.*

Taking Theorems 4, 5 and [4, Th. 2] into account we get some quasicontinuous selection theorems for multivalued maps with closed values.

We remind that a function $f\colon X \to Y$, where X, Y are topological spaces, is called quasicontinuous if for each open set $V \subset Y$, the set $f^{-1}(V)$ is semi-open [8].

COROLLARY 3. *Let X be a regular locally compact and paracompact space and let Y be a topological space, metrizable by a complete metric. If $F\colon X \to Y$ is a lower semicontinuous multivalued map with closed values, then F has a quasicontinuous selection.*

COROLLARY 4. *Let (X,d) be a dense in itself metric space with $E(X)$ discrete and let Y be a topological space, metrizable by a complete metric. If $F\colon X \to Y$ is a lower semicontinuous multivalued map with closed values, then F has a quasicontinuous selection.*

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