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RARE AND MEAGER SETS IN RELATOR SPACES

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ABSTRACT. Rare and meager subsets of relator space are investigated. In particular, necessary and sufficient conditions the fat set be nonmeager are given.

Since relator spaces are proper generalizations of not only uniform but also topological and closure spaces, the results obtained naturally extend, improve and supplement the corresponding results of former authors.

Introduction

A nonvoid family $\mathcal{R}$ of binary relations on a nonvoid set X is called a relator on X, and the ordered pair $X(\mathcal{R}) = (X, \mathcal{R})$ is called a relator space. Relator spaces are straightforward generalizations of ordered sets and uniform spaces.

Thus, in a relator space we can naturally define several order theoretic and topological notions. For instance, we can define the nearness of one set to another, and the convergence of one net to another. However, these are not needed now.

If $\mathcal{R}$ is a relator on X and $A \subset X$, then the members of the families

$$\operatorname{int}_{\mathcal{R}}(A) = \big\{x \in X\colon\ \exists\, R \in \mathcal{R}\colon R(x) \subset A\big\}$$

and

$$\operatorname{cl}_{\mathcal{R}}(A) = \big\{x \in X\colon\ \forall\, R \in \mathcal{R}\colon R(x) \cap A \neq \emptyset\big\}$$

are called the interior and adherence points of the set A in the relator space $X(\mathcal{R})$, respectively.

Moreover, the members of the families

$$\mathcal{T}_{\mathcal{R}} = \{A \subset X\colon A \subset \operatorname{int}_{\mathcal{R}}(A)\} \quad \text{and} \quad \mathcal{F}_{\mathcal{R}} = \big\{A \subset X\colon \operatorname{cl}_{\mathcal{R}}(A) \subset A\big\}$$

are called the open and closed subsets of the relator space $X(\mathcal{R})$, respectively.

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While, the members of the families

$$\mathcal{E}_{\mathcal{R}} = \{A \subset X : \operatorname{int}_{\mathcal{R}}(A) \neq \emptyset\} \quad \text{and} \quad \mathcal{D}_{\mathcal{R}} = \{A \subset X : \operatorname{cl}_{\mathcal{R}}(A) = X\}$$

are called the fat and dense subsets of the relator space $X(\mathcal{R})$, respectively.

And the members of the families

$$\mathcal{N}_{\mathcal{R}} = \{A \subset X : \operatorname{cl}_{\mathcal{R}}(A) \notin \mathcal{E}_{\mathcal{R}}\} \quad \text{and} \quad \mathcal{M}_{\mathcal{R}} = \left\{ \bigcup_{n=1}^{\infty} A_n : A_n \in \mathcal{N}_{\mathcal{R}} \right\}$$

are called the rare and meager subsets of the relator space $X(\mathcal{R})$, respectively.

The relations $\operatorname{int}_{\mathcal{R}}$ and $\operatorname{cl}_{\mathcal{R}}$, and the families $\mathcal{T}_{\mathcal{R}}$, $\mathcal{F}_{\mathcal{R}}$, $\mathcal{E}_{\mathcal{R}}$ and $\mathcal{D}_{\mathcal{R}}$ have been intensively studied in several former papers. In particular, it turned out that the family $\mathcal{E}_{\mathcal{R}}$ is, in general, a more important tool than the family $\mathcal{T}_{\mathcal{R}}$.

However, the families $\mathcal{N}_{\mathcal{R}}$ and $\mathcal{M}_{\mathcal{R}}$, for an arbitrary relator $\mathcal{R}$, have not been studied earlier. Therefore, our main purpose here is to initiate a systematic study of the latter families. In particular, we shall give some necessary and sufficient conditions in order that $\mathcal{R}$ be a Baire relator in the sense that $\mathcal{E}_{\mathcal{R}} \cap \mathcal{M}_{\mathcal{R}} = \emptyset$.

1. A few basic facts on relations and relators

A subset F of a product set $X \times Y$ is called a relation on X to Y. In particular, the relations $\Delta_X = \{(x, x) : x \in X\}$ and $X^2 = X \times X$ are called the identity and the universal relations on X, respectively.

Namely, if in particular $X = Y$, then we may simply say that F is a relation on X. Note that if F is a relation on X to Y, then F is also a relation on $X \cup Y$. Therefore, it is frequently not a severe restriction to assume that $X = Y$.

If F is a relation on X and, moreover, $x \in X$ and $A \subset X$, then the sets $F(x) = \{y \in X : (x, y) \in F\}$ and $F[A] = \bigcup_{x \in A} F(x)$ are called the images of x and A under F, respectively. Whenever $A \in X$ seems unlikely, we may write $F(A)$ in place of $F[A]$.

If F is a relation on X, then the values $F(x)$, where $x \in X$, uniquely determine F. Therefore, the inverse F^{-1} of F can be defined such that $F^{-1}(x) = \{y \in X : x \in F(y)\}$ for all $x \in X$. Moreover, if G is also a relation on X, then the composition $G \circ F$ of G and F can be defined such that $(G \circ F)(x) = G(F(x))$ for all $x \in X$.

A relation f on X is called a function if for each $x \in D_f = f^{-1}(X)$ there exists $y \in X$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we usually write $f(x) = y$. Moreover, sometimes we also write $(f_x)_{x \in D_f} = f$ and $\{f_x\}_{x \in D_f} = f(X)$, where $f_x = f(x)$.

A relation R on X is called reflexive, symmetric, transitive, and directed if $\Delta_X \subset R$, $R = R^{-1}$, $R \circ R \subset R$, and $X^2 = R^{-1} \circ R$, respectively. Moreover, a reflexive relation is called a preorder (tolerance) if it is transitive (symmetric), and a directed preorder is called a direction.

A nonvoid family $\mathcal{R}$ of relations on a nonvoid set X is called a *relator* on X, and the ordered pair $X(\mathcal{R}) = (X, \mathcal{R})$ is called a *relator space*. Relator spaces are straightforward generalizations of the various ordered sets and uniform spaces [12]. They deserve to be widely investigated because of the following two facts.

If $\mathcal{D}$ is a nonvoid family of certain distance functions on X, then the relator $\mathcal{R}_{\mathcal{D}}$ consisting of all surroundings $B_r^d = \big\{(x,y) \in X^2 \colon d(x,y) < r\big\}$, where $d \in \mathcal{D}$ and $r > 0$, is a more convenient means of defining the basic notions of analysis in the space $X(\mathcal{D})$ than the family of all open subsets of $X(\mathcal{D})$, or even the family $\mathcal{D}$ itself.

Moreover, all reasonable generalizations of the usual topological structures (such as proximities, closures, topologies, filters and convergences, for instance) can be easily derived from relators (according to the results of [15] and [11]), and thus they need not be studied separately.

For instance, if $\mathcal{A}$ is a certain generalized topology or a nonvoid stack (ascending system) in X, then $\mathcal{A}$ can be easily derived (according to the forthcoming definitions of the families $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{E}_{\mathcal{R}}$) from the Davis–Pervin relator $\mathcal{R}_{\mathcal{A}}$ consisting of all preorders $R_A = A^2 \cup A^c \times X$, where $A \in \mathcal{A}$ and $A^c = X \setminus A$.

Note that, in contrast to the preorders R_A, the surroundings B_r^d are usually tolerances. Therefore, besides preorder relators, tolerance relators are also important particular cases of reflexive relators. Unfortunately, the class of all reflexive relators turned out to be inadequate for some important purposes.

In the sequel, we shall be frequently dealing with relations on families of subsets of a fixed set X. For instance, for $A \subset X$, we shall write

$$\mathcal{C}(A) = X \setminus A \quad \text{and} \quad \mathcal{P}(A) = \{B \colon B \subset A\}.$$

Note that thus $\mathcal{C}$ is a function and $\mathcal{P}$ is a relation on $\mathcal{P}(X)$ such that $\mathcal{C}^{-1}(A) = \mathcal{C}(A)$ and $\mathcal{P}^{-1}(A) = \{B \colon A \subset B \subset X\}$ for all $A \subset X$.

Moreover, for $A \subset X$ and $\mathcal{A} \subset \mathcal{P}(X)$, we shall also write

$$A^c = \mathcal{C}(A) \quad \text{and} \quad \mathcal{A}^c = \{A^c \colon A \in \mathcal{A}\}.$$

Note that this elementwise complement $\mathcal{A}^c$ of $\mathcal{A}$ should be carefully distinguished from the global complement $\mathcal{P}(X) \setminus \mathcal{A}$ of $\mathcal{A}$ which was also to be denoted by $\mathcal{A}^c$.

In the sequel, for $\mathcal{A} \subset \mathcal{P}(X)$, we shall also write

$$\mathcal{A}_* = \{B \subset X \colon \exists\, A \in \mathcal{A} \colon B \subset A\} \quad \text{and} \quad \mathcal{A}^* = \{B \subset X \colon \exists\, A \in \mathcal{A} \colon A \subset B\}.$$

Note that thus we have $\mathcal{A}_* = \mathcal{P}[\mathcal{A}]$ and $\mathcal{A}^* = \mathcal{P}^{-1}[\mathcal{A}]$. Therefore, the lower and upper stars are very particular closure operators on $\mathcal{P}(X)$. Moreover, we have

$$\mathcal{A}_* = \left((\mathcal{A}^c)^*\right)^c \quad \text{and} \quad \mathcal{A}^* = \left((\mathcal{A}^c)_*\right)^c.$$

Moreover, for $\mathcal{A} \subset \mathcal{P}(X)$, we shall also write

$$\mathcal{A}^\delta = \left\{ \bigcap_{n=1}^{\infty} A_n : \forall\, n \in \mathbb{N} :\, A_n \in \mathcal{A} \right\} \quad \text{and} \quad \mathcal{A}^\sigma = \left\{ \bigcup_{n=1}^{\infty} A_n : \forall\, n \in \mathbb{N} :\, A_n \in \mathcal{A} \right\}.$$

Note that thus δ and σ are also very particular closure operators on $\mathcal{P}(X)$. Moreover, we have

$$\mathcal{A}^\delta = \left((\mathcal{A}^c)^\sigma\right)^c \quad \text{and} \quad \mathcal{A}^\sigma = \left((\mathcal{A}^c)^\delta\right)^c,$$

and also $(\mathcal{A}_*)^\delta = (\mathcal{A}^\delta)_*$, $(\mathcal{A}_*)^\sigma = (\mathcal{A}^\sigma)_*$, $(\mathcal{A}^*)^\delta = (\mathcal{A}^\delta)^*$, $(\mathcal{A}^*)^\sigma = (\mathcal{A}^\sigma)^*$.

2. Some important basic tools in relator spaces

Definition 2.1. If $\mathcal{R}$ is a relator on X, and $\operatorname{int}_{\mathcal{R}}$ and $\operatorname{cl}_{\mathcal{R}}$ are relations on $\mathcal{P}(X)$ to X such that for all $A \subset X$

$$\operatorname{int}_{\mathcal{R}}(A) = \left\{ x \in X :\ \exists\, R \in \mathcal{R} :\ R(x) \subset A \right\}$$

and

$$\operatorname{cl}_{\mathcal{R}}(A) = \left\{ x \in X :\ \forall\, R \in \mathcal{R} :\ R(x) \cap A \neq \emptyset \right\},$$

then the members of the families $\operatorname{int}_{\mathcal{R}}(A)$ and $\operatorname{cl}_{\mathcal{R}}(A)$ are called the *interior* and *adherence* points of the set A in the relator space $X(\mathcal{R})$, respectively.

Theorem 2.2. *If $\mathcal{R}$ is a relator on X, then*

(1) $\operatorname{int}_{\mathcal{R}} = (\operatorname{cl}_{\mathcal{R}} \circ \mathcal{C})^c$;

(2) $cl_{\mathcal{R}} = (\operatorname{int}_{\mathcal{R}} \circ \mathcal{C})^c$.

Hint. Note that the assertion (2) in a detailed form means only that $\operatorname{cl}_{\mathcal{R}}(A) = X \setminus \operatorname{int}_{\mathcal{R}}(X \setminus A)$ for all $A \subset X$.

Theorem 2.3. *If $\mathcal{R}$ is a relator on X, then*

(1) $\operatorname{int}_{\mathcal{R}}(X) = X$;

(2) $\operatorname{int}_{\mathcal{R}}(A) \subset \operatorname{int}_{\mathcal{R}}(B)$ *if* $A \subset B \subset X$.

Theorem 2.4. *If $\mathcal{R}$ is a relator on X, then*

(1) $\operatorname{cl}_{\mathcal{R}}(\emptyset) = \emptyset$;

(2) $\operatorname{cl}_{\mathcal{R}}(A) \subset \operatorname{cl}_{\mathcal{R}}(B)$ *if* $A \subset B \subset X$.

Theorem 2.5. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then*

$$\mathrm{cl}_{\mathcal{R}}(A) = \bigcap_{R \in \mathcal{R}} R^{-1}(A) \quad \textit{and} \quad \mathrm{int}_{\mathcal{R}}(A) = \bigcup_{R \in \mathcal{R}} R^{-1}(A^c)^c .$$

Remark 2.6. In [15], we have also proved that if int is a relation on $\mathcal{P}(X)$ to X such that $\mathrm{int}(X) = X$ and $\mathrm{int}(A) \subset \mathrm{int}(B)$ if $A \subset B \subset X$, then $\mathrm{int} = \mathrm{int}_{\mathcal{R}_{\mathrm{int}}}$, where $\mathcal{R}_{\mathrm{int}} = \big\{ R_{(x,A)} \colon x \in \mathrm{int}(A) \big\}$ and $R_{(x,A)} = \{x\} \times A \cup \{x\}^c \times X$.

Definition 2.7. If $\mathcal{R}$ is a relator on X, then the members of the families

$$\mathcal{T}_{\mathcal{R}} = \big\{ A \subset X \colon A \subset \mathrm{int}_{\mathcal{R}}(A) \big\} \quad \text{and} \quad \mathcal{F}_{\mathcal{R}} = \big\{ A \subset X \colon \mathrm{cl}_{\mathcal{R}}(A) \subset A \big\}$$

are called the *open* and *closed subsets* of the relator space $X(\mathcal{R})$, respectively.

Theorem 2.8. *If $\mathcal{R}$ is a relator on X, then*

$$\mathcal{T}_{\mathcal{R}} = \big\{ A \subset X \colon \forall x \in A \colon \exists R \in \mathcal{R} \colon R(x) \subset A \big\}$$

and

$$\mathcal{F}_{\mathcal{R}} = \big\{ A \subset X \colon \forall x \in A^c \colon \exists R \in \mathcal{R} \colon R(x) \cap A = \emptyset \big\} .$$

Theorem 2.9. *If $\mathcal{R}$ is a relator on X, then*

(1) $\mathcal{T}_{\mathcal{R}} = (\mathcal{F}_{\mathcal{R}})^c$;

(2) $\mathcal{F}_{\mathcal{R}} = (\mathcal{T}_{\mathcal{R}})^c$.

Hint. Note that the assertion (2) in a detailed form means only that $\mathcal{F}_{\mathcal{R}} = \{A \subset X \colon X \setminus A \in \mathcal{T}_{\mathcal{R}}\}$.

Theorem 2.10. *If $\mathcal{R}$ is a relator on X, then*

(1) $X \in \mathcal{T}_{\mathcal{R}}$;

(2) $\bigcup \mathcal{A} \in \mathcal{T}_{\mathcal{R}}$ *for all* $\mathcal{A} \subset \mathcal{T}_{\mathcal{R}}$.

Theorem 2.11. *If $\mathcal{R}$ is a relator on X, then*

(1) $\emptyset \in \mathcal{F}_{\mathcal{R}}$;

(2) $\bigcap \mathcal{A} \in \mathcal{F}_{\mathcal{R}}$ *for all* $\mathcal{A} \subset \mathcal{F}_{\mathcal{R}}$.

Remark 2.12. In [15], we have also proved that if $\mathcal{T} \subset \mathcal{P}(X)$ such that $X \in \mathcal{T}$ and $\bigcup \mathcal{A} \in \mathcal{T}$ for all $\mathcal{A} \subset \mathcal{T}$, then $\mathcal{T} = \mathcal{T}_{\mathcal{R}_{\mathcal{T}}}$, where $\mathcal{R}_{\mathcal{T}}$ is the Davis–Pervin relator generated by $\mathcal{T}$.

Definition 2.13. If $\mathcal{R}$ is a relator on X, then the members of the families

$$\mathcal{E}_{\mathcal{R}} = \{ A \subset X \colon \mathrm{int}_{\mathcal{R}}(A) \neq \emptyset \} \quad \text{and} \quad \mathcal{D}_{\mathcal{R}} = \{ A \subset X \colon \mathrm{cl}_{\mathcal{R}}(A) = X \}$$

are called the *fat* and *dense subsets* of the relator space $X(\mathcal{R})$, respectively.

Theorem 2.14. *If $\mathcal{R}$ is a relator on X, then*

$$\mathcal{E}_{\mathcal{R}} = \{R(x) \colon x \in X\,,\ R \in \mathcal{R}\}^*$$

and

$$\mathcal{D}_{\mathcal{R}} = \big\{A \subset X \colon \forall R \in \mathcal{R} \colon R^{-1}(A) = X\big\}\,.$$

Theorem 2.15. *If $\mathcal{R}$ is a relator on X, then*

(1) $\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\} \subset \mathcal{E}_{\mathcal{R}}$;

(2) $\mathcal{F}_{\mathcal{R}} \cap \mathcal{D}_{\mathcal{R}} \subset \{X\}$.

Theorem 2.16. *If $\mathcal{R}$ is a relator on X, then*

(1) $\mathcal{E}_{\mathcal{R}} = \big(\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}}\big)^c$;

(2) $\mathcal{D}_{\mathcal{R}} = \big(\mathcal{P}(X) \setminus \mathcal{E}_{\mathcal{R}}\big)^c$.

Hint. Note that the assertion (2) in a detailed form means only that $\mathcal{D}_{\mathcal{R}} = \{A \subset X \colon X \setminus A \notin \mathcal{E}_{\mathcal{R}}\}$.

Theorem 2.17. *If $\mathcal{R}$ is a relator on X, then*

(1) $\mathcal{E}_{\mathcal{R}} = \{A \subset X \colon \forall B \in \mathcal{D}_{\mathcal{R}} \colon A \cap B \neq \emptyset\}$;

(2) $\mathcal{D}_{\mathcal{R}} = \{A \subset X \colon \forall B \in \mathcal{E}_{\mathcal{R}} \colon A \cap B \neq \emptyset\}$.

Theorem 2.18. *If $\mathcal{R}$ is a relator on X, then*

(1) $X \in \mathcal{E}_{\mathcal{R}}$;

(2) $\mathcal{E}_{\mathcal{R}} = (\mathcal{E}_{\mathcal{R}})^*$.

Theorem 2.19. *If $\mathcal{R}$ is a relator on X, then*

(1) $\emptyset \notin \mathcal{D}_{\mathcal{R}}$;

(2) $\mathcal{D}_{\mathcal{R}} = (\mathcal{D}_{\mathcal{R}})^*$.

Remark 2.20. In [15], we have also proved that if $\mathcal{E} \subset \mathcal{P}(X)$ such that $X \in \mathcal{E}$ and $\mathcal{E} = \mathcal{E}^*$, then $\mathcal{E} = \mathcal{E}_{\mathcal{R}_{\mathcal{E}}}$, where $\mathcal{R}_{\mathcal{E}}$ is the Davis–Pervin relator generated by $\mathcal{E}$.

3. Some relevant classifications of relators

Definition 3.1. If $\mathcal{R}$ is a relator on X, then we say that:

(1) $\mathcal{R}$ is *reflexive* if $x \in R(x)$ for all $x \in X$ and $R \in \mathcal{R}$;

(2) $\mathcal{R}$ is *quasi-topological* if $x \in \operatorname{int}_{\mathcal{R}}\Big(\operatorname{int}_{\mathcal{R}}\big(R(x)\big)\Big)$ for all $x \in X$ and $R \in \mathcal{R}$;

(3) $\mathcal{R}$ is *topological* if for all $x \in X$ and $R \in \mathcal{R}$ there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $x \in V \subset R(x)$.

Remark 3.2. Note that the inclusion $x \in \operatorname{int}_{\mathcal{R}}\big(R(x)\big)$ trivially holds for all $x \in X$ and $R \in \mathcal{R}$.

The appropriateness of Definition 3.1 is quite obvious from the following three basic theorems which have been mostly proved in [14].

Theorem 3.3. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is reflexive;*

(2) *$\bigcap \mathcal{R}$ is reflexive;*

(3) *$\operatorname{int}_{\mathcal{R}}(A) \subset A$ for all $A \subset X$;*

(4) *$A \subset \operatorname{cl}_{\mathcal{R}}(A)$ for all $A \subset X$.*

Theorem 3.4. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is quasi-topological;*

(2) *$\operatorname{int}_{\mathcal{R}}\big(R(x)\big) \in \mathcal{T}_{\mathcal{R}}$ for all $x \in X$ and $R \in \mathcal{R}$;*

(3) *$\operatorname{int}_{\mathcal{R}}(A) \in \mathcal{T}_{\mathcal{R}}$ for all $A \subset X$;*

(4) *$\operatorname{cl}_{\mathcal{R}}(A) \in \mathcal{F}_{\mathcal{R}}$ for all $A \subset X$.*

Theorem 3.5. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is topological;*

(2) *$\mathcal{R}$ is reflexive and quasi-topological;*

(3) *$\operatorname{int}_{\mathcal{R}}(A) = \bigcup\{V \in \mathcal{T}_{\mathcal{R}} : V \subset A\}$ for all $A \subset X$;*

(4) *$\operatorname{cl}_{\mathcal{R}}(A) = \bigcap\{W \in \mathcal{F}_{\mathcal{R}} : A \subset W\}$ for all $A \subset X$.*

Remark 3.6. By Theorem 3.4, a relator $\mathcal{R}$ on X may be called weakly (strongly) quasi-topological if $\operatorname{cl}_{\mathcal{R}}\big(\{x\}\big) \in \mathcal{F}_{\mathcal{R}}$ for all $x \in X$ ($R(x) \in \mathcal{T}_{\mathcal{R}}$ for all $x \in X$ and $R \in \mathcal{R}$).

Moreover, by Theorem 3.5, the relator $\mathcal{R}$ may be called weakly (strongly) topological if it is reflexive and weakly (strongly) quasi-topological.

Definition 3.7. If $\mathcal{R}$ is a relator on X, then we say that:

(1) $\mathcal{R}$ is *total* if $R(x) \neq \emptyset$ for all $x \in X$ and $R \in \mathcal{R}$;

(2) $\mathcal{R}$ is *quasi-directed* if $R(x) \cap S(y) \in \mathcal{E}_{\mathcal{R}}$ for all $x, y \in X$ and $R, S \in \mathcal{R}$;

(3) $\mathcal{R}$ is *topologically filtered* if $x \in \operatorname{int}_{\mathcal{R}}\big(R(x) \cap S(x)\big)$ for all $x \in X$ and $R, S \in \mathcal{R}$.

Remark 3.8. Whenever $A \subset X$ such that $R(x) \neq \emptyset$ for all $x \in A$ and $R \in \mathcal{R}$, then we also say that the relator $\mathcal{R}$ is total on A.

Moreover, if $R(x) \cap R(y) \neq \emptyset$ for all $x, y \in X$ and $R, S \in \mathcal{R}$, then we say that the relator $\mathcal{R}$ is semi-directed. Thus, a semi-directed relator is total and a total quasi-directed relator is semi-directed.

The appropriateness of Definition 3.7 is quite obvious from the following three basic theorems which have been mostly proved in [14].

Theorem 3.9. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is total;*

(2) $\emptyset \notin \mathcal{E}_{\mathcal{R}}$;

(3) $\mathcal{D}_{\mathcal{R}} \neq \emptyset$;

(4) $X \in \mathcal{D}_{\mathcal{R}}$;

(5) $\mathcal{E}_{\mathcal{R}} \neq \mathcal{P}(X)$.

Theorem 3.10. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is quasi-directed;*

(2) $A \cap B \in \mathcal{E}_{\mathcal{R}}$ *for all* $A, B \in \mathcal{E}_{\mathcal{R}}$;

(3) $A \cap B \in \mathcal{D}_{\mathcal{R}}$ *for all* $A \in \mathcal{E}_{\mathcal{R}}$ *and* $B \in \mathcal{D}_{\mathcal{R}}$;

(4) $A \cup B \in \mathcal{D}_{\mathcal{R}}$ *implies that* $A \in \mathcal{D}_{\mathcal{R}}$ *or* $B \in \mathcal{D}_{\mathcal{R}}$.

Theorem 3.11. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is topologically filtered;*

(2) $\mathrm{cl}_{\mathcal{R}}(A \cup B) = \mathrm{cl}_{\mathcal{R}}(A) \cup \mathrm{cl}_{\mathcal{R}}(B)$ *for all* $A, B \subset X$;

(3) $\mathrm{int}_{\mathcal{R}}(A \cap B) = \mathrm{int}_{\mathcal{R}}(A) \cap \mathrm{int}_{\mathcal{R}}(B)$ *for all* $A, B \subset X$.

Corollary 3.12. *If $\mathcal{R}$ is a topologically filtered relator on X, then the families $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{F}_{\mathcal{R}}$ are closed under finite intersections and unions, respectively.*

Remark 3.13. Note that, by Theorem 3.5, for a topological relator $\mathcal{R}$ on X the converse statement is also true.

4. Some further properties of the preceding basic tools

A particular case of the first statement of the following theorem has also been observed in the proof of a theorem of Janković, Ganster and Reilly [6].

Theorem 4.1. *If $\mathcal{R}$ is a reflexive relator on X and $A \subset X$, then*

(1) $\operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A) \setminus A\big) = \emptyset$;

(2) $\operatorname{int}_{\mathcal{R}}\big(A \setminus \operatorname{int}_{\mathcal{R}}(A)\big) = \emptyset$.

P r o o f . Since the assertion (1) can be easily derived from the assertion (2) by using Theorem 2.2, we shall only prove the assertion (2).

For this, note that if $x \in \operatorname{int}_{\mathcal{R}}\big(A \setminus \operatorname{int}_{\mathcal{R}}(A)\big)$, then by the definition of $\operatorname{int}_{\mathcal{R}}$ there exists $R \in \mathcal{R}$ such that $R(x) \subset A \setminus \operatorname{int}_{\mathcal{R}}(A)$. This implies that $R(x) \subset A$ and $R(x) \cap \operatorname{int}_{\mathcal{R}}(A) = \emptyset$. Hence, by the definition of $\operatorname{int}_{\mathcal{R}}$ and the reflexivity of $\mathcal{R}$, it follows that $x \in \operatorname{int}_{\mathcal{R}}(A)$ and $x \notin \operatorname{int}_{\mathcal{R}}(A)$. This contradiction proves the assertion (2). □

Now, as an immediate consequence of the above theorem and the corresponding definitions, we can also state

Corollary 4.2. *If $\mathcal{R}$ is a reflexive relator on X and $A \subset X$, then*

(1) $A \in \mathcal{F}_{\mathcal{R}} \iff \operatorname{cl}_{\mathcal{R}}(A) \setminus A \in \mathcal{T}_{\mathcal{R}}$;

(2) $A \in \mathcal{T}_{\mathcal{R}} \iff A \setminus \operatorname{int}_{\mathcal{R}}(A) \in \mathcal{T}_{\mathcal{R}}$.

A particular case of the first statement of the following theorem has also been observed in the proof of the assertion (9.1.7) of C s á s z á r [1, p. 385]. It is a natural extension of the essential part of Theorem 1.3.6 of E n g e l k i n g [3, p. 44].

Theorem 4.3. *If $\mathcal{R}$ is a topologically filtered relator on X and $A \subset X$, then*

(1) $V \subset \operatorname{cl}_{\mathcal{R}}(A)$ *implies* $V \subset \operatorname{cl}_{\mathcal{R}}(A \cap V)$ *for all* $V \in \mathcal{T}_{\mathcal{R}}$;

(2) $\operatorname{int}_{\mathcal{R}}(A) \subset W$ *implies* $\operatorname{int}_{\mathcal{R}}(A \cup W) \subset W$ *for all* $W \in \mathcal{F}_{\mathcal{R}}$.

P r o o f . Since the assertion (1) can be easily derived from the assertion (2) by using Theorems 2.2 and 2.9, we shall only prove the assertion (2).

For this, suppose on the contrary that there exists $W \in \mathcal{F}_{\mathcal{R}}$ such that $\operatorname{int}_{\mathcal{R}}(A) \subset W$, but $\operatorname{int}_{\mathcal{R}}(A \cup W) \not\subset W$. Then, there exists $x \in X$ such that $x \in \operatorname{int}_{\mathcal{R}}(A \cup W)$, but $x \notin W$. Hence, by the definition of $\operatorname{int}_{\mathcal{R}}$, it follows that there exists $R \in \mathcal{R}$ such that $R(x) \subset A \cup W$. Moreover, since $\operatorname{cl}_{\mathcal{R}}(W) \subset W$, we also have $x \notin \operatorname{cl}_{\mathcal{R}}(W)$. Hence, by the definition of $cl_{\mathcal{R}}$, it follows that there exists $S \in \mathcal{R}$ such that $S(x) \cap W = \emptyset$. Moreover, since $\mathcal{R}$ is topologically directed, there exists $T \in \mathcal{R}$ such that $T(x) \subset R(x) \cap S(x)$. Hence, we can already see that $T(x) \subset A \cup W$ and $T(x) \cap W = \emptyset$. Therefore, $T(x) \subset A$, and thus $x \in \operatorname{int}_{\mathcal{R}}(A) \subset W$. This contradiction proves the assertion (2). □

Now, as an immediate consequence of the above theorem and the corresponding definitions, we can also state

Corollary 4.4. *If $\mathcal{R}$ is a topologically filtered relator on X and $A \subset X$, then*

(1) $V \subset \operatorname{cl}_{\mathcal{R}}(A)$ *implies* $V \subset A$ *for all* $V \in \mathcal{T}_{\mathcal{R}}$ *with* $A \cap V \in \mathcal{F}_{\mathcal{R}}$;

(2) $\operatorname{int}_{\mathcal{R}}(A) \subset W$ *implies* $A \subset W$ *for all* $W \in \mathcal{F}_{\mathcal{R}}$ *with* $A \cup W \in \mathcal{T}_{\mathcal{R}}$.

The relationship of the fat sets to the open sets in a topological relator space is quite straightforward by the following

Theorem 4.5. *If $\mathcal{R}$ is a topological relator on X and $A \subset X$, then*

(1) $A \in \mathcal{E}_{\mathcal{R}}$ *if and only if there exists* $V \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$ *such that* $V \subset A$;

(2) $A \in \mathcal{D}_{\mathcal{R}}$ *if and only if for all* $W \in \mathcal{F}_{\mathcal{R}} \setminus \{X\}$ *we have* $A \setminus W \neq \emptyset$.

Proof. If $A \in \mathcal{E}_{\mathcal{R}}$ and $V = \operatorname{int}_{\mathcal{R}}(A)$, then by the definition of $\mathcal{E}_{\mathcal{R}}$ we have $V \neq \emptyset$. Moreover, by Theorems 3.3 and 3.4, we also have $V \subset A$ and $V \in \mathcal{T}_{\mathcal{R}}$.

Conversely, if $V \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$ such that $V \subset A$, then by the definition of $\mathcal{T}_{\mathcal{R}}$ and Theorem 2.3, we have $\emptyset \neq V \subset \operatorname{int}_{\mathcal{R}}(V) \subset \operatorname{int}_{\mathcal{R}}(A)$. Therefore, by the definition of $\mathcal{E}_{\mathcal{R}}$, we also have $A \in \mathcal{E}_{\mathcal{R}}$.

Therefore, the assertion (1) is true. The assertion (2) can now be easily derived from the assertion (1) by using Theorems 2.9 and 2.16. □

Remark 4.6. The assertion (1) of the above theorem can be briefly expressed by saying that the family $\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$ is a base for the stack $\mathcal{E}_{\mathcal{R}}$ in the sense that $\mathcal{E}_{\mathcal{R}} = \big(\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}\big)^*$.

Unfortunately, if the latter statement holds for a relator $\mathcal{R}$ on X, then instead of the topologicalness of $\mathcal{R}$ we can only state that $\operatorname{int}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}\big(R(x)\big)\big) \neq \emptyset$, and hence $\operatorname{int}_{\mathcal{R}}\big(R(x)\big) \in \mathcal{E}_{\mathcal{R}}$ for all $x \in X$ and $R \in \mathcal{R}$.

5. Rare sets in relator spaces

Definition 5.1. If $\mathcal{R}$ is a relator on X, then the members of the family

$$\mathcal{N}_{\mathcal{R}} = \big\{A \subset X :\ \operatorname{cl}_{\mathcal{R}}(A) \notin \mathcal{E}_{\mathcal{R}}\big\}$$

are called the *rare subsets* of the relator space $X(\mathcal{R})$.

Remark 5.2. By the corresponding definitions, for any $A \subset X$ we have $A \in \mathcal{N}_{\mathcal{R}}$ if and only if $\operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A)\big) = \emptyset$.

Moreover, by Theorems 2.16 and 2.2, it is clear that we also have the following

Theorem 5.3. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $A \in \mathcal{N}_{\mathcal{R}}$;

(2) $\mathrm{cl}_{\mathcal{R}}(A)^c \in \mathcal{D}_{\mathcal{R}}$;

(3) $\mathrm{int}_{\mathcal{R}}(A^c) \in \mathcal{D}_{\mathcal{R}}$.

Unfortunately, the relationship of the rare sets to the open sets is not so straightforward. As a counterpart of an assertion of Kuratowski [7, p. 37], we can only prove the following

Theorem 5.4. *If $\mathcal{R}$ is a topologically filtered, topological relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $A \in \mathcal{N}_{\mathcal{R}}$;

(2) *for each $U \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$ there exists $V \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$ such that $V \subset U$ and $A \cap V = \emptyset$.*

Proof. If the assertion (1) holds, then by Theorem 5.3 we have $\mathrm{int}_{\mathcal{R}}(A^c) \in \mathcal{D}_{\mathcal{R}}$. Moreover, by Theorem 3.4, we also have $\mathrm{int}_{\mathcal{R}}(A^c) \in \mathcal{T}_{\mathcal{R}}$. Therefore, if $U \in \mathcal{T}_{\mathcal{R}}$ and $V = U \cap \mathrm{int}_{\mathcal{R}}(A^c)$, then by Corollary 5.12 we have $V \in \mathcal{T}_{\mathcal{R}}$. Moreover, if in addition $U \neq \emptyset$, then by Theorem 2.15 we also have $U \in \mathcal{E}_{R}$. Therefore, by Theorem 2.17, we also have $V \neq \emptyset$. Moreover, by the definition of V, it is clear that $V \subset U$ and $V \subset \mathrm{int}_{\mathcal{R}}(A^c)$. Hence, by Theorem 3.3, it follows that $V \subset A^c$, and thus $A \cap V = \emptyset$. Therefore, the assertion (2) also holds.

To prove the converse implication (2)$\Rightarrow$(1), suppose now, on the contrary, that the assertion (2) holds, but the assertion (1) does not hold. Then, by Remark 5.2, we have $\mathrm{int}_{\mathcal{R}}\big(\mathrm{cl}_{\mathcal{R}}(A)\big) \neq \emptyset$. Moreover, by Theorem 3.4 we also have $\mathrm{int}_{\mathcal{R}}\big(\mathrm{cl}_{\mathcal{R}}(A)\big) \in \mathcal{T}_{\mathcal{R}}$. Therefore, by the assertion (2), there exists $V \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$ such that $V \subset \mathrm{int}_{\mathcal{R}}\big(\mathrm{cl}_{\mathcal{R}}(A)\big)$ and $A \cap V = \emptyset$, i.e., $A \subset V^c$. Hence, by Theorem 3.2, it follows that $V \subset \mathrm{cl}_{\mathcal{R}}(A)$. Moreover, by Theorem 2.4 and 2.9, it follows that $\mathrm{cl}_{\mathcal{R}}(A) \subset \mathrm{cl}_{\mathcal{R}}(V^c) \subset V^c$. Therefore, $V \subset V^c$, and hence $V = \emptyset$. This contradiction proves the required implication. □

From Theorem 5.4, by using Theorem 5.5, we can easily get the following

Theorem 5.5. *If $\mathcal{R}$ is a topologically filtered, topological relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $A \in \mathcal{N}_{\mathcal{R}}$;

(2) *for each $U \in \mathcal{E}_{\mathcal{R}}$ there exists $V \in \mathcal{E}_{\mathcal{R}}$ such that $V \subset U$ and $A \cap V = \emptyset$.*

In view of the latter theorem, it is also worth proving the following

Theorem 5.6. *If $\mathcal{R}$ is a quasi-directed relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $A \in \mathcal{N}_{\mathcal{R}}$;

(2) *for each $U \in \mathcal{E}_{\mathcal{R}}$ there exists $V \in \mathcal{D}_{\mathcal{R}}$ such that $V \subset U$ and $V \cap \mathrm{cl}_{\mathcal{R}}(A) = \emptyset$.*

P r o o f . If the assertion (1) holds, then by Theorem 5.3 we have $\mathrm{cl}_{\mathcal{R}}(A)^c \in \mathcal{D}_{\mathcal{R}}$. Therefore, if $U \in \mathcal{E}_{\mathcal{R}}$ and $V = U \cap \mathrm{cl}_{\mathcal{R}}(A)^c$, then by Theorem 3.10 we have $V \in \mathcal{D}_{\mathcal{R}}$. Moreover, it is clear that $V \subset U$ and $V \subset \mathrm{cl}_{\mathcal{R}}(A)^c$, and hence $V \cap \mathrm{cl}_{\mathcal{R}}(A) = \emptyset$. Therefore, the assertion (2) also holds.

To prove the converse implication (2)$\Rightarrow$(1), suppose now on the contrary that the assertion (2) holds, but the assertion (1) does not hold. Then, by Definition 5.1, we have $\mathrm{cl}_{\mathcal{R}}(A) \in \mathcal{E}_{\mathcal{R}}$. Therefore, by the assertion (2), there exists $V \in \mathcal{D}_{\mathcal{R}}$ such that $V \subset cl_{\mathcal{R}}(A)$ and $V \cap \mathrm{cl}_{\mathcal{R}}(A) = \emptyset$. Hence, it follows that $V = \emptyset$, and thus $\emptyset \in \mathcal{D}_{\mathcal{R}}$. This contradiction proves the required implication. □

Remark 5.7. Note that the implications (2)$\Rightarrow$(1) in Theorems 5.4, 5.5 and 5.6 do not require the relator $\mathcal{R}$ to be topologically filtered and quasi-directed, respectively.

For topological relators, instead of Remark 5.2, we can also use the following

THEOREM 5.8. *If $\mathcal{R}$ is a topological relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $A \in \mathcal{N}_{\mathcal{R}}$;

(2) $A \cap \mathrm{int}_{\mathcal{R}}\big(\mathrm{cl}_{\mathcal{R}}(A)\big) = \emptyset$.

P r o o f . If the assertion (1) holds, then by Remark 5.2 it is clear that the assertion (2) also holds.

To prove the converse implication, note that if the assertion (1) does not hold, then by Remark 5.2 there exists $x \in X$ such that $x \in \mathrm{int}_{\mathcal{R}}\big(\mathrm{cl}_{\mathcal{R}}(A)\big)$. Hence, by Theorems 3.4 and 2.8, it follows that there exists $R \in \mathcal{R}$ such that

$$R(x) \subset \mathrm{int}_{\mathcal{R}}\big(\mathrm{cl}_{\mathcal{R}}(A)\big) .$$

This, by the reflexivity of R and Theorem 3.3, implies that $x \in \mathrm{cl}_{\mathcal{R}}(A)$. Hence, by Definition 2.1, it follows that $A \cap R(x) \neq \emptyset$. Therefore, by the above inclusion, the assertion (2) does not also hold. □

Now, as a sharpening of the incorrectly proved statement (9.1.7) of C s á s z á r [1, p. 385], we can also prove the following theorem.

THEOREM 5.9. *If $\mathcal{R}$ is a topologically filtered, topological relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $A \in \mathcal{N}_{\mathcal{R}}$;

(2) *for each $x \in A \cap \mathrm{int}_{\mathcal{R}}\big(\mathrm{cl}_{\mathcal{R}}(A)\big)$ there exists $U \in \mathcal{T}_{\mathcal{R}}$ such that $x \in U$ and $A \cap U \in \mathcal{N}_{\mathcal{R}}$.*

P r o o f . If the assertion (1) holds, then since $X \in \mathcal{T}_{\mathcal{R}}$ and $A \cap X = A$, it is clear that the assertion (2) trivially holds.

To prove the converse implication, suppose now on the contrary that the assertion (2) holds, but the assertion (1) does not hold. Then, by Theorem 5.8, there exists $x \in X$ such that $x \in A \cap \operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A)\big)$. Therefore, by the assertion (2), there exists $U \in \mathcal{T}_{\mathcal{R}}$ such that $x \in U$ and $A \cap U \in \mathcal{N}_{\mathcal{R}}$. Define $V = U \cap \operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A)\big)$. Then, we evidently have $x \in V$ and $V \subset U$. Moreover, by Theorem 3.4, Corollary 3.12 and Theorem 3.3, it is clear that we also have

$$V \in \mathcal{T}_{\mathcal{R}} \qquad \text{and} \qquad V \subset \operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A)\big) \subset \operatorname{cl}_{\mathcal{R}}(A)\,.$$

Hence, by Theorem 4.3, it follows that $V \subset \operatorname{cl}_{\mathcal{R}}(A \cap V)$. Therefore, by Definition 2.7 and Theorems 2.3 and 2.4, we also have

$$x \in V \subset \operatorname{int}_{\mathcal{R}}(V) \subset \operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A \cap V)\big) \subset \operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A \cap U)\big)\,.$$

Hence, by Remark 5.2, it follows that $A \cap U \notin \mathcal{N}_{\mathcal{R}}$. This contradiction proves the required implication. □

6. Some further properties of rare sets

Theorem 6.1. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\emptyset \in \mathcal{N}_{\mathcal{R}}$;

(2) *$\mathcal{R}$ is total.*

P r o o f . By Theorems 5.3 and 3.9, it is clear that

$$(1) \iff \operatorname{int}_{\mathcal{R}}(\emptyset^c) \in \mathcal{D}_{\mathcal{R}} \iff \operatorname{int}_{\mathcal{R}}(X) \in \mathcal{D}_{\mathcal{R}} \iff (2)\,.$$

□

Theorem 6.2. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $X \notin \mathcal{N}_{\mathcal{R}}$;

(2) *$\mathcal{R}$ is total on some $A \in \mathcal{E}_{\mathcal{R}}$.*

P r o o f . If the assertion (1) holds and $A = \operatorname{cl}_{\mathcal{R}}(X)$, then by Definition 5.1 we have $A \in \mathcal{E}_{\mathcal{R}}$. Moreover, by Theorem 2.5, we also have $A = \bigcap_{R \in \mathcal{R}} R^{-1}(X)$. This implies that $A \subset R^{-1}(X)$ for all $R \in \mathcal{R}$. Hence, we can infer that $R(x) = R(x) \cap X \neq \emptyset$ for all $x \in A$ and $R \in \mathcal{R}$. Therefore, the assertion (2) also holds.

The converse, implication (2) ⇒ (1) can be proved quite similarly by reversing the above argument. □

Corollary 6.3. *If $\mathcal{R}$ is a total relator on X, then $X \notin \mathcal{N}_{\mathcal{R}}$.*

P r o o f . By Theorem 2.10, we have $X \in \mathcal{E}_{\mathcal{R}}$. Therefore, Theorem 6.2 can be applied. □

Theorem 6.4. *If $\mathcal{R}$ is a relator on X, then*

(1) $\mathcal{D}_{\mathcal{R}} \cap \mathcal{N}_{\mathcal{R}} = \emptyset$;

(2) $\mathcal{N}_{\mathcal{R}} = (\mathcal{N}_{\mathcal{R}})_*$.

P r o o f . If $A \in \mathcal{D}_{\mathcal{R}}$, then by the Definition 2.13 and Theorem 2.18, we have $\mathrm{cl}_{\mathcal{R}}(A) = X \in \mathcal{E}_{\mathcal{R}}$. Therefore, by Definition 5.1, we also have $A \notin \mathcal{N}_{\mathcal{R}}$.

While, if $A \subset B \in \mathcal{N}_{\mathcal{R}}$, then by Theorem 2.4 and Definition 6.1 we have $\mathrm{cl}_{\mathcal{R}}(A) \subset \mathrm{cl}_{\mathcal{R}}(B) \notin \mathcal{E}_{\mathcal{R}}$. Hence, by Theorem 2.18 (2), it follows that $\mathrm{cl}_{\mathcal{R}}(A) \notin \mathcal{E}_{\mathcal{R}}$, and thus by Definition 5.1 we also have $A \in \mathcal{N}_{\mathcal{R}}$. □

Theorem 6.5. *If $\mathcal{R}$ is a reflexive relator on X, then*

(1) $\mathcal{E}_{\mathcal{R}} \cap \mathcal{N}_{\mathcal{R}} = \emptyset$;

(2) $\mathrm{cl}_{\mathcal{R}}(A) \in \mathcal{N}_{\mathcal{R}}$ *implies* $A \in \mathcal{N}_{\mathcal{R}}$.

P r o o f . If $A \in \mathcal{N}_{\mathcal{R}}$, then by Theorem 3.3 and Definition 5.1 we have $A \subset \mathrm{cl}_{\mathcal{R}}(A) \notin \mathcal{E}_{\mathcal{R}}$. Hence, by Theorem 2.18 (2), it is clear that $A \notin \mathcal{E}_{\mathcal{R}}$.

On the other hand, if $A \subset X$ such that $\mathrm{cl}_{\mathcal{R}}(A) \in \mathcal{N}_{\mathcal{R}}$, then by the inclusion $A \subset \mathrm{cl}_{\mathcal{R}}(A)$ and Theorem 6.4 (2), it is clear that $A \in \mathcal{N}_{\mathcal{R}}$. □

Theorem 6.6. *If $\mathcal{R}$ is a quasi-topological relator on X, then $A \in \mathcal{N}_{\mathcal{R}}$ implies $\mathrm{cl}_{\mathcal{R}}(A) \in \mathcal{N}_{\mathcal{R}}$.*

P r o o f . If $A \in \mathcal{N}_{\mathcal{R}}$, then by Definition 5.1 we have $\mathrm{cl}_{\mathcal{R}}(A) \notin \mathcal{E}_{\mathcal{R}}$. Moreover, by Theorem 3.4, we also have $\mathrm{cl}_{\mathcal{R}}(A) \in \mathcal{F}_{\mathcal{R}}$, and thus $\mathrm{cl}_{\mathcal{R}}\big(\mathrm{cl}_{R}(A)\big) \subset \mathrm{cl}_{\mathcal{R}}(A)$. Hence, by Theorem 2.18 (2), it is clear that $\mathrm{cl}_{\mathcal{R}}\big(\mathrm{cl}_{R}(A)\big) \notin \mathcal{E}_{\mathcal{R}}$. Therefore, by Definition 5.1, we also have that $\mathrm{cl}_{\mathcal{R}}(A) \in \mathcal{N}_{\mathcal{R}}$ is also true. □

Now, as an immediate consequence of Theorems 3.5, 6.5 and 6.6, we can also state

Corollary 6.7. *If $\mathcal{R}$ is a topological relator on X, then for any $A \subset X$ we have $A \in \mathcal{N}_{\mathcal{R}}$ if and only if $\mathrm{cl}_{\mathcal{R}}(A) \in \mathcal{N}_{\mathcal{R}}$.*

The first statement of the following theorem is an extension of an assertion of K u r a t o w s k i [7, p. 37].

Theorem 6.8. *If $\mathcal{R}$ is a topologically filtered, quasi-topological relator on X and $A \subset X$, then*

(1) $A \notin \mathcal{E}_{\mathcal{R}}$ *implies* $A \cup \mathrm{cl}_{\mathcal{R}}(B) \notin \mathcal{E}_{\mathcal{R}}$ *for all* $B \in \mathcal{N}_{\mathcal{R}}$;

(2) $A \in \mathcal{D}_{\mathcal{R}}$ *implies* $A \setminus \mathrm{cl}_{\mathcal{R}}(B) \in \mathcal{D}_{\mathcal{R}}$ *for all* $B \in \mathcal{N}_{\mathcal{R}}$.

P r o o f . Suppose that $A \notin \mathcal{E}_{\mathcal{R}}$ and $B \in \mathcal{N}_{\mathcal{R}}$. Then, by Definition 2.13, we have $\operatorname{int}_{\mathcal{R}}(A) = \emptyset$, and thus $\operatorname{int}_{\mathcal{R}}(A) \subset \operatorname{cl}_{\mathcal{R}}(B)$. Moreover, from Theorem 3.4, we know that $\operatorname{cl}_{\mathcal{R}}(B) \in \mathcal{F}_{\mathcal{R}}$. Therefore, by Theorem 4.3, we also have

$$\operatorname{int}_{\mathcal{R}}\big(A \cup \operatorname{cl}_{\mathcal{R}}(B)\big) \subset \operatorname{cl}_{\mathcal{R}}(B).$$

Hence, by Theorems 3.4 and 2.3 and Remark 6.2, it is already clear that

$$\operatorname{int}_{\mathcal{R}}\big(A \cup \operatorname{cl}_{\mathcal{R}}(B)\big) \subset \operatorname{int}\big(\operatorname{int}_{\mathcal{R}}\big(A \cup \operatorname{cl}_{\mathcal{R}}(B)\big)\big) \subset \operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(B)\big) = \emptyset.$$

Therefore, by Definition 2.13, we also have $A \cup \operatorname{cl}_{\mathcal{R}}(B) \notin \mathcal{E}_{\mathcal{R}}$.

Thus, we have proved the assertion (1). The assertion (2) can be now easily derived from the assertion (1) by using Theorem 2.16. □

Now, as a counterpart of an assertion attributed to S. J a n i s z e w s k i by K u r a t o w s k i [7, p. 37], we can also prove the following

Theorem 6.9. *If $\mathcal{R}$ is a topologically filtered, quasi-topological relator on X, then $\mathcal{N}_{\mathcal{R}}$ is closed under finite unions.*

P r o o f . If $A, B \in \mathcal{N}_{\mathcal{R}}$, then by Definition 5.1 and Theorem 6.6, we also have $\operatorname{cl}_{\mathcal{R}}(A) \notin \mathcal{E}_{\mathcal{R}}$ and $\operatorname{cl}_{\mathcal{R}}(B) \in \mathcal{N}_{\mathcal{R}}$. Hence, by Theorem 3.11 and 6.8, it follows that

$$\operatorname{cl}_{\mathcal{R}}(A \cup B) = \operatorname{cl}_{\mathcal{R}}(A) \cup \operatorname{cl}_{\mathcal{R}}(B) \notin \mathcal{E}_{\mathcal{R}}.$$

Therefore, by Definition 5.1, we also have $A \cup B \in \mathcal{N}_{\mathcal{R}}$. Hence, by induction, it is clear that the required assertion is also true. □

Remark 6.10. The assertions 6.4 (2) and 6.9 can be briefly expressed by saying that $\mathcal{N}_{\mathcal{R}}$ is an ideal in $\mathcal{P}(X)$.

7. Meager sets in relator spaces

Definition 7.1. If $\mathcal{R}$ is a relator on X, then the members of the family

$$\mathcal{M}_{\mathcal{R}} = (\mathcal{N}_{\mathcal{R}})^{\sigma}$$

are called the *meager subsets* of the relator space $X(\mathcal{R})$.

Remark 7.2. By the corresponding definitions, for any $A \subset X$ we have $A \in \mathcal{M}_{\mathcal{R}}$ if and only if for each $n \in \mathbb{N}$ there $\exists A_n \in \mathcal{N}_{\mathcal{R}}$ such that $A = \bigcup_{n=1}^{\infty} A_n$.

THEOREM 7.3. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\emptyset \in \mathcal{M}_{\mathcal{R}}$;

(2) *$\mathcal{R}$ is total.*

Proof. By Remark 7.2, it is clear that $\emptyset \in \mathcal{M}_{\mathcal{R}}$ if and only if $\emptyset \in \mathcal{N}_{\mathcal{R}}$. Therefore, by Theorem 6.1, the assertions (1) and (2) are also equivalent. □

By Remark 7.2, Theorem 6.4 (2) and Definition 7.1, it is clear that we also have

THEOREM 7.4. *If $\mathcal{R}$ is a relator on X, then*

(1) $\mathcal{N}_{\mathcal{R}} \subset \mathcal{M}_{\mathcal{R}}$;

(2) $\mathcal{M}_{\mathcal{R}} = (\mathcal{M}_{\mathcal{R}})_*$;

(3) $\mathcal{M}_{\mathcal{R}} = (\mathcal{M}_{\mathcal{R}})^{\sigma}$.

Remark 7.5. The assertions (2) and (3) can be briefly expressed by saying that $\mathcal{M}_{\mathcal{R}}$ is a σ-ideal in $\mathcal{P}(X)$.

THEOREM 7.6. *If $\mathcal{R}$ is a relator on X, then*

$$(\mathcal{M}_{\mathcal{R}})^c \subset (\mathcal{E}_{\mathcal{R}})^{\delta}.$$

Proof. By Theorem 6.4, we have

$$\mathcal{D}_{\mathcal{R}} \cap \mathcal{N}_{\mathcal{R}} = \emptyset, \quad \text{and thus} \quad \mathcal{N}_{\mathcal{R}} \subset \mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}}.$$

Hence, by Definition 7.1, the corresponding properties of the operations σ and δ, and Theorem 3.16, it is clear that

$$(\mathcal{M}_{\mathcal{R}})^c = \big((\mathcal{N}_{\mathcal{R}})^{\sigma}\big)^c = \big((\mathcal{N}_{\mathcal{R}})^c\big)^{\delta} \subset \big((\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}})^c\big)^{\delta} = (\mathcal{E}_{\mathcal{R}})^{\delta}.$$

□

From the above argument, by Theorem 6.5(1), it is clear that we also have

THEOREM 7.7. *If $\mathcal{R}$ is a reflexive relator on X, then*

$$(\mathcal{M}_{\mathcal{R}})^c \subset (\mathcal{D}_{\mathcal{R}})^{\delta}.$$

In view of the fact that $\mathcal{E}_{\mathcal{R}} \cap \mathcal{N}_{\mathcal{R}} = \emptyset$ whenever $\mathcal{R}$ is a reflexive relator, we may naturally introduce the following extension of a usual definition of a Baire space [4, p. 11].

Definition 7.8. A relator $\mathcal{R}$ on X is called a *Baire relator* if

$$\mathcal{E}_{\mathcal{R}} \cap \mathcal{M}_{\mathcal{R}} = \emptyset\,.$$

Remark 7.9. Note that thus a relator space $X(\mathcal{R})$ is a Baire relator space if and only if the fat subsets of $X(\mathcal{R})$ are not meager, or equivalently the meager subsets of $X(\mathcal{R})$ are not fat.

Theorem 7.10. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is a Baire relator;*

(2) $(\mathcal{M}_{\mathcal{R}})^c \subset \mathcal{D}_{\mathcal{R}}$.

Proof. By the corresponding definitions and Theorem 2.16, we evidently have

$$(1) \iff \mathcal{M}_{\mathcal{R}} \subset \mathcal{P}(X) \setminus \mathcal{E}_{\mathcal{R}} \iff (\mathcal{M}_{\mathcal{R}})^c \subset \big(\mathcal{P}(X) \setminus \mathcal{E}_{\mathcal{R}}\big)^c \iff (2)\,.$$

□

Remark 7.11. As an immediate consequence of Definition 7.8 and Remark 4.6, we can state at once that a topological relator $\mathcal{R}$ is a Baire relator if and only if $\mathcal{M}_{\mathcal{R}} \cap \big(\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}\big)^* = \emptyset$.

However, it is now more interesting to note that we can also prove the following

Theorem 7.12. *If $\mathcal{R}$ is a topological relator on X, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is a Baire relator;*

(2) $\mathcal{T}_{\mathcal{R}} \cap \mathcal{M}_{\mathcal{R}} = \{\emptyset\}$;

(3) $\mathcal{F}_{\mathcal{R}} \cap (\mathcal{M}_{\mathcal{R}})^c = \{X\}$.

Proof. Now, by Theorems 2.10 and 7.3, we have $\{\emptyset\} \subset \mathcal{T}_{\mathcal{R}} \cap \mathcal{M}_{\mathcal{R}}$. Therefore, if the assertion (2) does not hold, then there exists $A \in \mathcal{T}_{\mathcal{R}} \cap \mathcal{M}_{\mathcal{R}}$ such that $A \notin \{\emptyset\}$. This implies that $A \in \big(\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}\big) \cap \mathcal{M}_{\mathcal{R}}$. Hence, by Theorem 2.15, it follows that $A \in \mathcal{E}_{\mathcal{R}} \cap \mathcal{M}_{\mathcal{R}}$. Therefore, the assertion (1) does not hold either.

Conversely, if the assertion (1) does not hold, then there exists $A \subset X$ such that $A \in \mathcal{E}_{\mathcal{R}} \cap \mathcal{M}_{\mathcal{R}}$. This implies that $A \in \mathcal{E}_{\mathcal{R}}$ and $A \in \mathcal{M}_{\mathcal{R}}$. Hence, by Theorem 4.4, it is clear that there exists $V \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$ such that $V \subset A$. Moreover, by Theorem 7.4 (2), it is clear that we also have $V \in \mathcal{M}_{\mathcal{R}}$. Hence, we can already see that $V \in \mathcal{T}_{\mathcal{R}} \cap \mathcal{M}_{\mathcal{R}}$ such that $V \notin \{\emptyset\}$. Therefore, the assertion (2) does not hold either.

Thus, we have proved that the assertions (1) and (2) are equivalent. Moreover, by using Theorem 2.9, we can easily see that the assertions (2) and (3) are also equivalent. □

8. Some further characterizations of Baire relators

The following theorem is a straightforward extension of a simple but important characterization of Baire topological spaces.

Theorem 8.1. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is a Baire relator;*

(2) *if $\bigcup_{i=1}^{\infty} A_n \in \mathcal{E}_{\mathcal{R}}$, then $\mathrm{cl}_{\mathcal{R}}(A_n) \in \mathcal{E}_{\mathcal{R}}$ for some $n \in \mathbb{N}$;*

(3) *if $\mathrm{int}_{\mathcal{R}}(A_n) \in \mathcal{D}_{\mathcal{R}}$ for all $n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{D}_{\mathcal{R}}$.*

P r o o f. Suppose that the assertion (1) holds, and $A_n \subset X$ and $\mathrm{cl}_{\mathcal{R}}(A_n) \notin \mathcal{E}_{\mathcal{R}}$ for all $n \in \mathbb{N}$. Then, by the corresponding definitions, we have $A_n \in \mathcal{N}_{\mathcal{R}}$ for all $n \in \mathbb{N}$, and thus $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}_{\mathcal{R}}$. Hence, by the assertion (1), it follows that $\bigcup_{n=1}^{\infty} A_n \notin \mathcal{E}_{\mathcal{R}}$. Therefore, the assertion (2) also holds.

Suppose now that the assertion (2) holds and $A \in \mathcal{M}_{\mathcal{R}}$. Then, by Remark 6.2, for each $n \in \mathbb{N}$, there exists $A_n \in \mathcal{N}_{\mathcal{R}}$ such that $A = \bigcup_{n=1}^{\infty} A_n$. Hence, by Definition 5.1, it follows that $\mathrm{cl}_{\mathcal{R}}(A_n) \notin \mathcal{E}_{\mathcal{R}}$ for all $n \in \mathbb{N}$. Hence, by the assertion (2), it is clear that $A \notin \mathcal{E}_{\mathcal{R}}$. Therefore, the assertion (1) also holds. Finally, to prove the equivalence of the assertions (2) and (3), we note that by Theorems 2.2 and 2.16

$$\mathrm{int}_{\mathcal{R}}(A_n) \in \mathcal{D}_{\mathcal{R}} \iff \mathrm{cl}_{\mathcal{R}}(A_n^c)^c \in \mathcal{D}_{\mathcal{R}} \iff \mathrm{cl}_{\mathcal{R}}(A_n^c) \notin \mathcal{E}_{\mathcal{R}}.$$

Moreover, $\bigcap_{n=1}^{\infty} A_n \in \mathcal{D}_{\mathcal{R}}$ if and only if $\bigcup_{n=1}^{\infty} A_n^c \notin \mathcal{E}_{\mathcal{R}}$. □

Now, as a more familiar particular case of the above theorem, we can also state

Corollary 8.2. *If $\mathcal{R}$ is a topological relator on X, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is a Baire relator;*

(2) *$(\mathcal{T}_{\mathcal{R}} \cap \mathcal{D}_{\mathcal{R}})^{\delta} \subset \mathcal{D}_{\mathcal{R}}$;*

(3) *$\mathcal{E}_{\mathcal{R}} \cap (\mathcal{F}_{\mathcal{R}} \setminus \mathcal{E}_{\mathcal{R}})^{\sigma} = \emptyset$.*

P r o o f. To prove the implication (1)⇒(2), note that if $A \in (\mathcal{T}_{\mathcal{R}} \cap \mathcal{D}_{\mathcal{R}})^{\delta}$, then by the definition of the operation δ for each $n \in \mathbb{N}$ there exists $A_n \in$

$\mathcal{T}_{\mathcal{R}} \cap \mathcal{D}_{\mathcal{R}}$ such that $A = \bigcap_{n=1}^{\infty} A_n$. Hence, by the corresponding definitions, it follows that $A_n \subset \operatorname{int}_{\mathcal{R}}(A_n)$ and $A_n \in \mathcal{D}_{\mathcal{R}}$ for all $n \in \mathbb{N}$. Thus, by Theorem 2.19 (2), we also have $\operatorname{int}_{\mathcal{R}}(A_n) \in \mathcal{D}_{\mathcal{R}}$ for all $n \in \mathbb{N}$. Therefore, if the assertion (1) holds, then by Theorem 8.1, we necessarily have $A = \bigcap_{n=1}^{\infty} A_n \in \mathcal{D}_{\mathcal{R}}$. Thus, the assertion (2) also holds.

To prove the converse implication (2)$\Rightarrow$(1), note that if for each $n \in \mathbb{N}$ we have $A_n \subset X$ such that $\operatorname{int}_{\mathcal{R}}(A_n) \in \mathcal{D}_{\mathcal{R}}$, then by Theorem 3.4 we also have $\operatorname{int}_{\mathcal{R}}(A_n) \in \mathcal{T}_{\mathcal{R}} \cap \mathcal{D}_{\mathcal{R}}$. Therefore, if the assertion (2) holds, then we necessarily have $\bigcap_{n=1}^{\infty} \operatorname{int}_{\mathcal{R}}(A_n) \in \mathcal{D}_{\mathcal{R}}$. Hence, by Theorems 2.3 and 2.19 (2), it is clear that $\bigcap_{n=1}^{\infty} A_n \in \mathcal{D}_{\mathcal{R}}$. Therefore, by Theorem 8.1, the assertion (1) also holds.

Finally, to prove the equivalence of the assertions (2) and (3), we note that by Theorems 2.2 and 2.16 and the corresponding properties of the operations c, δ and σ we have

$$(\mathcal{T}_{\mathcal{R}} \cap \mathcal{D}_{\mathcal{R}})^{\delta} = \big((\mathcal{F}_{\mathcal{R}})^{c} \cap \big(\mathcal{P}(X) \setminus \mathcal{E}_{\mathcal{R}}\big)^{c}\big)^{\delta} = \big((\mathcal{F}_{\mathcal{R}} \setminus \mathcal{E}_{\mathcal{R}})^{c}\big)^{\delta} = \big((\mathcal{F}_{\mathcal{R}} \setminus \mathcal{E}_{\mathcal{R}})^{\sigma}\big)^{c}$$

and $\mathcal{D}_{\mathcal{R}} = \big(\mathcal{P}(X) \setminus \mathcal{E}_{\mathcal{R}}\big)^{c}$. Therefore, the assertion (2) holds if and only if $(\mathcal{F}_{\mathcal{R}} \setminus \mathcal{E}_{\mathcal{R}})^{\sigma} \subset \mathcal{P}(X) \setminus \mathcal{E}_{\mathcal{R}}$, that is, the assertion (3) holds. $\square$

The following theorem is counterpart of a recent theorem of Janković, Ganster and Reilly [6] whose origin goes again back to Kuratowski [7, p. 323].

Theorem 8.3. *If $\mathcal{R}$ is a topologically directed, topological relator on X, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is a Baire relator;*

(2) $\mathcal{M}_{\mathcal{R}} \cap (\mathcal{T}_{\mathcal{R}})^{\delta} \subset \mathcal{N}_{\mathcal{R}}$;

(3) $(\mathcal{M}_{\mathcal{R}})^{c} \cap (\mathcal{F}_{\mathcal{R}})^{\sigma} \subset (\mathcal{N}_{\mathcal{R}})^{c}$.

Proof. Suppose that the assertion (1) holds and $A \in \mathcal{M}_{\mathcal{R}} \cap (\mathcal{T}_{\mathcal{R}})^{\delta}$. Then, $A \in \mathcal{M}_{\mathcal{R}}$ and $A \in (\mathcal{T}_{\mathcal{R}})^{\delta}$. Therefore, for each $n \in \mathbb{N}$ there exists $V_n \in \mathcal{T}_{\mathcal{R}}$ such that $A = \bigcap_{n=1}^{\infty} V_n$. Hence, it follows that

$$\operatorname{cl}_{\mathcal{R}}(A) \setminus A = \operatorname{cl}_{\mathcal{R}}(A) \setminus \bigcap_{n=1}^{\infty} V_n = \bigcup_{n=1}^{\infty} \big(\operatorname{cl}_{\mathcal{R}}(A) \setminus V_n\big) = \bigcup_{n=1}^{\infty} \operatorname{cl}_{\mathcal{R}}(A) \cap V_n^{c}.$$

Define $W_n = \operatorname{cl}_{\mathcal{R}}(A) \cap V_n^{c}$ for all $n \in \mathbb{N}$. Then, by Theorems 3.4 and 2.9 and Corollary 3.12, it is clear that $W_n \in \mathcal{F}_{\mathcal{R}}$, and thus $\operatorname{cl}_{\mathcal{R}}(W_n) \subset W_n$ for all $n \in \mathbb{N}$.

Moreover, it is clear that

$$\mathrm{cl}_{\mathcal{R}}(A) \setminus A = \bigcup_{n=1}^{\infty} W_n$$

and thus $W_n \subset \mathrm{cl}_{\mathcal{R}}(A) \setminus A$ for all $n \in \mathbb{N}$. Hence, by using Theorems 2.3 and 4.1, we can already see that

$$\mathrm{int}_{\mathcal{R}}\big(\mathrm{cl}_{\mathcal{R}}(W_n)\big) \subset \mathrm{int}_{\mathcal{R}}(W_n) \subset \mathrm{int}_{\mathcal{R}}\big(\mathrm{cl}_{\mathcal{R}}(A) \setminus A\big) = \emptyset\,,$$

and thus $W_n \in \mathcal{N}_{\mathcal{R}}$ for all $n \in \mathbb{N}$. Therefore, $\mathrm{cl}_{\mathcal{R}}(A) \setminus A \in \mathcal{M}_{\mathcal{R}}$, and thus by Theorems 3.3 and 7.4 (3)

$$\mathrm{cl}_{\mathcal{R}}(A) = A \cup \big(\mathrm{cl}_{\mathcal{R}}(A) \setminus A\big) \in \mathcal{M}_{\mathcal{R}}\,.$$

Hence, by the assertion (1), it follows that $\mathrm{cl}_{\mathcal{R}}(A) \notin \mathcal{E}_{\mathcal{R}}$, and thus $A \in \mathcal{N}_{\mathcal{R}}$. Therefore, the assertion (2) also holds.

To prove the converse implication (2) $\Rightarrow$ (1), suppose now on the contrary that the assertion (2) holds, but the assertion (1) does not hold. Then there exists $A \in \mathcal{E}_{\mathcal{R}} \cap \mathcal{M}_{\mathcal{R}}$. This implies that $A \in \mathcal{E}_{\mathcal{R}}$ and $A \in \mathcal{M}_{\mathcal{R}}$. Thus, by Theorem 4.5, there exists $V \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$ such that $V \subset A$. Moreover, by Theorems 2.15 and Theorem 7.4(2), we also have $V \in \mathcal{E}_{\mathcal{R}}$ and $V \in \mathcal{M}_{\mathcal{R}}$. Now, by using the assertion (2), we can note that

$$V \in \mathcal{M}_{\mathcal{R}} \cap \mathcal{T}_{\mathcal{R}} \ \subset\ \mathcal{M}_{\mathcal{R}} \cap (\mathcal{T}_{\mathcal{R}})^{\delta} \ \subset\ \mathcal{N}_{\mathcal{R}}\,.$$

Therefore, $\mathrm{cl}_{\mathcal{R}}(V) \notin \mathcal{E}_{\mathcal{R}}$. Hence, by Theorems 3.3 and 2.18, it follows that $V \notin \mathcal{E}_{\mathcal{R}}$. This contradiction proves the required implication.

Finally, to prove the equivalence of the assertions (2) and (3), we note that by Theorem 2.9 and the corresponding property of the operations c, δ and σ we have

$$(\mathcal{T}_{\mathcal{R}})^{\delta} = \big((\mathcal{F}_{\mathcal{R}})^{c}\big)^{\delta} = \big((\mathcal{F}_{\mathcal{R}})^{\sigma}\big)^{c}.$$

Therefore, the assertion (2) holds if and only if the assertion (3) holds. □

Remark 8.4. Concerning the above characterizations of Baire relators, note that the members of the families $(\mathcal{T}_{\mathcal{R}})^{\delta}$ and $(\mathcal{M}_{\mathcal{R}})^{c}$ may be called the δ-open and the residual subsets of the relator spaces $X(\mathcal{R})$, respectively.

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