Tatra Mt. Math. Publ. **28** (2004), 43–56

TATRA MOUNTAINS
Mathematical Publications

A NOTE ON σ-DISCRETE AND MASSIVE SPACES

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ABSTRACT. Some cardinality properties of σ-discrete and massive spaces are considered. These spaces were essentially used in the study of the problem on the existence of ω-primitives [Z. Dusyński, Z. Grande and S. P. Ponomarev: *On the ω-primitive*, Math. Slovaca **51** (2001), 469–476], [J. Ewert and S. P. Ponomarev: *ω-primitives on σ-discrete metric spaces*, Tatra Mt. Math. Publ. **24** (2002), 13–37], [J. Ewert and S. P. Ponomarev: *On the existence of ω-primitives on arbitrary metric spaces*, Math. Slovaca **3** (2003), 51–57], [J. Ewert and S. P. Ponomarev: *Oscillation and ω-primitives*, Real Anal. Exchange **26** (2001–2002), 687–702]. It is shown, in particular, that unlike metric spaces, a topological space which is not locally σ-discrete, need not contain a massive subspace. Moreover, a locally σ-discrete topological space need not be σ-discrete, and a massive topological space may contain a discrete subset of cardinality greater than the remaining set (in contrast with the case of metric spaces). Some other cardinality properties of the mentioned spaces, in particular under GCH or MA, are also discussed.

1. Preliminaries

In this section we will give necessary definitions and formulate the basic problems.

DEFINITION 1. ([1]–[4]). A topological space $X = (X, \tau)$ is called

- *σ-discrete at $x \in X$* if there is a neighborhood U of x which is a σ-discrete subset of X, i.e.,
$$U = \bigcup_{n=1}^{\infty} A_n ,$$
where A_n are discrete sets;
- *locally σ-discrete* if X is σ-discrete at each of its points;
- *massive* if X is not σ-discrete at any $x \in X$.

Let us write "USC function" instead of "upper semicontinuous function".

2000 Mathematics Subject Classification: 26A15, 54C30.
Keywords: oscillation, ω-primitive, σ-discrete space, massive space, cardinal functions, Q-space.

Definition 2. ([1]) Let $f\colon X \to [0,\infty]$ be an USC function. A function $F\colon X \to \mathbb{R}$ is called an *ω-primitive* for f if

$$\forall x \in X: \quad \omega(F,x) = f(x)\,,$$

where $\omega(F,x)$ is the oscillation of F at x.

The main problem in the theory of ω-primitives could be formulated as follows. Does any USC function $f\colon X \to [0,\infty]$, vanishing at isolated points of X, have an ω-primitive? This problem was solved (and the answer is 'Yes') for all metric spaces [3] and for some particular class of nonmetrizable spaces [4]. It should be pointed out that the solutions to the problem given in [1]–[4] involved the use of σ-discrete spaces and massive spaces essentially. This is the main reason for considering these spaces in the present paper.

The second problem. Assume that f has an ω-primitive. Does there a Borel ω-primitive F for f exist and what could the minimal Baire class of F be? In the mentioned papers it was shown that for the considered spaces ω-primitives could be always found in at most Baire class 2. So far, neither of the problems have been solved in the general setting for nonmetrizable spaces.

Throughout the paper, we consider only spaces with infinitely many elements.

2. On a decomposition of a space into the union of σ-discrete and massive subspaces

Definition 3. A set $E \subset X = (X,\tau)$ is called *discrete*, *σ-discrete*, *locally σ-discrete*, *massive* if it is such as a subspace of X for the induced topology.

Theorem 1. (a) *Discreteness, σ-discreteness, local σ-discreteness, massiveness are invariant under homeomorphisms.*

(b) *Let $E \subset X = (X,\tau)$ be a massive set. Then each set A such that $E \subset A \subset X$ is neither σ-discrete nor locally σ-discrete.*

(c) *Let $X = (X,\tau)$ be a massive space. Then for any space Y the product space $X \times Y$ is massive.*

The proofs of (a), (b) are straightforward from the definitions. To prove (c), suppose not. Then there is a σ-discrete open set $U \times V \subset X \times Y$. Fix any $y \in V$. Since $U \subset X$ is massive, the set $U \times \{y\}$ is also massive as a set homeomorphic to U. However, then by (b), the set $U \times V$ is not σ-discrete, a contradiction.

Definition 4. ([3]) Given a topological space X, we set

$$\Sigma(X) = \{x \in X : X \text{ is } \sigma\text{-discrete at } x\}$$

and

$$\mathcal{M}(X) = X \setminus \Sigma(X).$$

It is immediate that $\Sigma(X)$ is a locally σ-discrete and open subset of X while $\mathcal{M}(X)$ is a closed subset of X. Question: is $\Sigma(X)$ σ-discrete and $\mathcal{M}(X)$ massive (whenever nonempty)? For metric spaces, the answer is 'Yes'. Namely, we have the following result.

Theorem 2. ([3]) (i) *Each locally σ-discrete metric space is σ-discrete.*

(ii) *Each metric space X can be written in the form*

$$X = \Sigma(X) \cup \mathcal{M}(X), \tag{1}$$

where $\Sigma(X)$ is σ-discrete and $\mathcal{M}(X)$ is massive, whenever nonempty.

This representation is obviously unique. The subsets $\Sigma(X)$, $\mathcal{M}(X)$ may be termed respectively as a "σ-discrete" and a "massive" part of X.

The following example shows that Theorem 2 fails in the case of nonmetrizable spaces.

Example 1. Let ω_1 be the first uncountable ordinal. Consider the classical space $X = \{\alpha\colon 0 \le \alpha \le \omega_1\}$ with the usual interval topology. The subspace $X_0 = \{\alpha\colon \alpha < \omega_1\}$ is locally σ-discrete because each interval (α, β), $\beta < \omega_1$, is at most countable. We claim that X_0 is not σ-discrete. Assume this is not the case. Then

$$X_0 = \bigcup_{n=1}^{\infty} X_n,$$

where each X_n is discrete. There exists X_m, $m \ge 1$, which is similar to X_0 (by a similarity mapping we mean an order preserving bijection). Indeed, otherwise, each X_n would be similar to an interval $\{\alpha\colon \alpha < \gamma_n < \omega_1\}$ which is countable. However, this is impossible since X_0 is uncountable. So let $f\colon X_m \to X_0$ be a similarity mapping. Let ω be the least countable ordinal. Since $f^{-1}(\omega)$ is an isolated point of X_m there is an interval $(\beta_1, \beta_2) \subset X_0$ such that $(\beta_1, \beta_2) \bigcap X_m = \{f^{-1}(\omega)\}$. It follows that $f(\beta_1) \in \mathbb{N}$ whence we get $f(\beta_1) < f(\beta_1) + 1 < \omega$. Therefore $\beta_1 < f^{-1}\big(f(\beta_1) + 1\big) < f^{-1}(\omega)$. This contradicts the fact that $(\beta_1, \beta_2) \bigcap X_m = \{f^{-1}(\omega)\}$. Thus X_0 is locally σ-discrete but not σ-discrete.

Now, observe that X is not σ-discrete at ω_1. Indeed, if there existed a σ-discrete neighborhood $(\alpha_0, \omega_1]$ of ω_1 the subspace X_0 would be σ-discrete because $(0, \alpha_0]$ is countable, a contradiction. It follows that $\Sigma(X) = X_0$ and $\mathcal{M}(X) = \{\omega_1\}$. Thus

$$X = \Sigma(X) \cup \mathcal{M}(X)$$

but $\Sigma(X)$ is not σ-discrete and $\mathcal{M}(X)$ is not massive.

Remark 1. Theorem 2 holds, in fact, for topological spaces admitting a σ-discrete base, because only this property was used in the proof of Theorem 2. Recall that spaces having a σ-discrete base need not be metrizable (cf., e.g., [5]). Open problem: describe classes of nonmetrizable spaces for which a corresponding analogue of Theorem 2 is true, i.e., properties (i), (ii) hold (without the adjective "metric").

3. Comparing the cardinalities for σ-discrete and massive spaces

We adopt the standard notations for cardinal functions (cf. [6]). In particular, $|X|$, $w(X)$, $s(X)$ stand respectively for the cardinality, weight and spread of X. Denote $\mathfrak{c} = |\mathbb{R}|$. Recall that we consider only infinite spaces.

Definition 5. ([1]) Let A be a nonempty subset of a metric space $X = (X, d)$. We let $\Delta A = \inf\{d(x,y) \colon x, y \in A\}$ if $|A| > 1$, and $\Delta A = \infty$ if $|A| = 1$.

It is clear that if $\Delta A > 0$ then A is closed and discrete.

Lemma 1. ([2], Lemma 1) Each σ-discrete metric space X can be written in the form

$$X = \bigcup_{n=1}^{\infty} A_n\,, \tag{2}$$

where $\Delta A_n > 0$ and $A_n \cap A_m = \emptyset$ for $n \neq m$.

Lemma 2. ([2], Lemma 3) Let $A_1, \ldots, A_n$ be pairwise disjoint subsets of a metric space X and $\Delta A_i > 0$, $1 \le i \le n$. Then there exists a family of pairwise disjoint open balls

$$\mathcal{B} = \big\{B\big(x, r(x)\big) : x \in A_1 \cup \cdots \cup A_n\big\}.$$

In other words, we may pick a ball around each $x \in A_1 \cup \cdots \cup A_n$ so that all these balls will be pairwise disjoint.

Theorem 3. *If X is a σ-discrete metric space then*

$$w(X) = |X|\,. \tag{3}$$

P r o o f. By Lemma 1, we may write (2) whence

$$|X| = \max\Big(\sup_n |A_n|\,,\, \omega\Big) = \sup_n \left|\bigcup_{i=1}^{n} A_i\right|. \tag{4}$$

Since each collection $\{A_1, \dots, A_n\}$, $n \in \mathbb{N}$, of the sets from (2) satisfies the assumptions of Lemma 2, we have that $\bigcup_{i=1}^{n} A_i$ is a discrete subset of X for each $n \in \mathbb{N}$. Therefore the right-hand part of (4) equals $s(X)$, i.e., spread of X, which equals $w(X)$. □

THEOREM 4. *Let D be a σ-discrete subset of a massive metric space X. Then*

$$|D| \leq |X \setminus D|. \tag{5}$$

Proof. By Lemma 1,

$$D = \bigcup_{n=1}^{\infty} D_n \,,$$

where D_n are pairwise disjoint sets, $\Delta D_n > 0$. Let $X_1 = X \setminus D$. Since X is massive and D σ-discrete, the set X_1 is massive. Consider the family of pairwise disjoint open balls

$$\mathcal{B}_1 = \big\{ B(x, r_1) \colon x \in D_1 \,,\ r_1 = \Delta D_1/3 \big\}. \tag{6}$$

Fix a mapping $f_1 \colon D_1 \to X_1$ such that $f_1(x) \in B(x, r_1) \setminus \{x\}$ for each $x \in D_1$. It is clear that such a mapping exists and is injective since $\mathcal{B}_1$ is composed of disjoint balls. It follows from (6) that $\Delta f_1(D_1) \geq r_1/3 > 0$, whence $f_1(D_1)$ is discrete. Therefore $X_2 = X_1 \setminus f_1(D_1)$ is massive and we may apply the previous argument for X_2 and D_2, we consider the family of open pairwise disjoint balls

$$\mathcal{B}_2 = \big\{ B(x, r_2) \colon x \in D_2 \,,\ r_2 = \Delta D_2/3 \big\}.$$

We then obtain an injective mapping $f_2 \colon D_2 \to X_2$, such that $\Delta f_2(D_2) \geq r_2/3$. Proceeding by induction, we obtain a sequence of injective mappings $\{f_n \colon D_n \to X_n \,,\ n \in \mathbb{N}\}$ with the properties:

(i) $\Delta f_n(D_n) > 0$;

(ii) $X_{n+1} = X_n \setminus f_n(D_n)$.

Now, define $f \colon D \to X \setminus D$ letting $f|D_n = f_n$. Since all the sets D_n, $f_m(D_m)$, $n, m \in \mathbb{N}$, are pairwise disjoint, the mapping f is injective, whence (5) follows. □

Remark 2. For metric spaces, the equality $|D| = |X \setminus D|$ is possible as the following example shows.

EXAMPLE 2. Consider the space X which is the union of $\mathfrak{c}$ copies $[0,1]_y$ of the closed interval $[0,1]$:

$$X = \bigcup_{y \in \mathbb{R}} [0,1]_y$$

with the metric

$$d(x_1, x_2) = \begin{cases} |x_1 - x_2| & \text{if } x_1, x_2 \in [0,1]_y\,, \\ 1 & \text{if } x \in [0,1]_{y_1},\ x_2 \in [0,1]_{y_2},\ y_1 \neq y_2\,. \end{cases} \tag{7}$$

Let D be any subset of X which has only one point in common with each $[0,1]_y$. Then we clearly have $|D| = \mathfrak{c} = |X \setminus D| = |X|$. Notice that the space X in this example is complete but not connected. To have an example of a connected and complete metric space with the required properties, the so-called "hedgehog space of spininess $\mathfrak{c}$" denoted by $J(\mathfrak{c})$ ([5]) may be taken. We omit a simple argument showing that there exists a discrete set $D \subset J(\mathfrak{c})$ such that $|D| = \mathfrak{c} = \bigl|J(\mathfrak{c}) \setminus D\bigr| = \bigl|J(\mathfrak{c})\bigr|$.

Next, we will show that in the case of massive nonmetrizable spaces it is possible that $|D| > |X \setminus D|$ where X may be of arbitrarily large cardinality.

THEOREM 5. *For each cardinal $\kappa > \mathfrak{c}$ there exists a massive space X, $|X| = 2^\kappa$, which contains a discrete subset D such that $|D| = 2^\kappa$, $|X \setminus D| = \kappa$, so we have $|D| > |X \setminus D|$.*

Proof. Let us fix a set A, $|A| = \kappa$, and take the Cantor cube $E = \{0,1\}^A$. We have $|E| = 2^\kappa$. Let S be a subset of E consisting of all functions $\varphi\colon A \to \{0,1\}$ such that the set $\bigl\{x \in A\colon \varphi(x) = 1\bigr\}$ is finite. The set S is dense in E since each element of the base for the topology of E contains an element of S. Furthermore, it follows from the definition of S that $|S| = |A| = \kappa$.

Let $\mathbb{R}^+$ be the set of nonnegative reals equipped with the standard subspace topology. Define a topology on $Y = E \times \mathbb{R}^+$ by introducing a fundamental system of neighborhoods $\mathcal{W} = \bigl\{W(z,t)\colon (z,t) \in Y\bigr\}$ as follows.

1) Neighborhoods of $(z,0) \in Y$ are sets of the form $W(z,0) = \bigl\{(z,0)\bigr\} \cup U(z) \times (0,\varepsilon)$ where $U(z)$ is an open neighborhood of $z \in E$, and $(0,\varepsilon) \subset \mathbb{R}^+$.

2) Neighborhoods of $(z,t) \in Y$, $t > 0$, are sets of the form $W(z,t) = U(z) \times (t-\varepsilon,\, t+\varepsilon)$, $0 < \varepsilon < t$.

It is easy to check that $\mathcal{W}$ is a fundamental system of neighborhoods on Y. Next observe that each of these neighborhoods contains an interval of the form $I = \{z\} \times (a,b)$, $z \in E$, $0 < a < b$, which is a massive subset of Y because (a,b) is a massive subset of $\mathbb{R}^+$. By Theorem 1(b), this implies that none of the neighborhoods $W(z,t)$ is σ-discrete and therefore Y is a massive space. Now, define the subspace $X \subset Y$ letting

$$X = S \times \mathbb{R}^+ \cup (E \setminus S) \times \{0\}\,.$$

We claim that X is a massive space. To see this, let $(z,t) \in S \times \mathbb{R}^+$. Then each neighborhood of (z,t) obviously contains an interval of type I described above.

Let $z \in E \setminus S$. Then $(z,0) \in X$ and each neighborhood of this point contains a neighborhood of the form

$$W(z,0) \cap X = \{(z,0)\} \cup \big(U(z) \cap S\big) \times (0,\varepsilon). \tag{8}$$

Since S is dense in E, we have $U(z) \cap S \neq \emptyset$, whence it follows that the set (8) contains an interval $I' = \{z'\} \times (0,\varepsilon)$, $z' \in U(z) \cap S$. Therefore by Theorem 1(b) the space X is massive.

Consider the set $D = (E \setminus S) \times \{0\}$. This is a discrete subset of X. Indeed, if $z \in E \setminus S$, no neighborhood (8) meets D except at the point $(z,0)$.

Finally, let us compare $|D|$ and $|X \setminus D|$. We have $|D| = |E \setminus S| = |E| = 2^\kappa$ because $|S| = \kappa < 2^\kappa = |E|$. On the other hand, $X \setminus D = S \times \mathbb{R}^+$ whence $|X \setminus D| = \max\big(|S|, \mathfrak{c}\big) = |S| = \kappa$ which completes the proof. Observe that X is nonmetrizable, which is immediate by Theorem 4. $\square$

Of course, there exist σ-discrete dense in themselves metric spaces of any cardinality $\kappa \geq \mathfrak{a}$. For instance, the product space $X \times \mathbb{Q}$, where X is σ-discrete, $|X| = \kappa$, and $\mathbb{Q}$ is taken with its usual metric. However each point of $X \times \mathbb{Q}$ has a countable neighborhood. The space constructed in the next proposition lacks this "flaw" whenever $\kappa > \aleph_0$.

THEOREM 6. *Given any infinite cardinal κ, there exists a σ-discrete dense in itself metric space X such that each of its nonempty open subsets has cardinality κ. Moreover, X can always be regarded as a subset of a Hilbert space.*

P r o o f. Fix a set A, $|A| = \kappa$. Let H be the set of all functions $x\colon A \to \mathbb{R}$ which may take nonzero values on finite sets only. Define the inner product on H letting $(x\,|\,y) = \sum\limits_{a \in A} x(a)\,y(a)$ and the corresponding norm $\|x\| = \sqrt{(x\,|\,x)}$. Let $E = \{\chi_a \colon a \in A\}$ be the set of characteristic functions of singletons. Obviously H is a real pre-Hilbert space [10] and E is an orthonormal set in H of cardinality κ. Put $E_0 = E$. Clearly, $\Delta E_0 = \sqrt{2}$. Fix any $r_1 \in (0,1)$ and let $R_1 = E_0 + r_1 E$, $E_1 = R_1 \cup E_0$ (by definition, $A_1 + \lambda A_2 = \{\xi_1 + \lambda\xi_2 \colon \xi_1 \in A_1,\ \xi_2 \in A_2\}$). Let $x, y \in R_1$, $x \neq y$. Then $x = x_0 + r_1 u_0$, $y = y_0 + r_1 v_0$ where $x_0, y_0, u_0, v_0 \in E_0$ which yields $\|x - y\| \geq \Delta E_0 - r_1\sqrt{2} = \sqrt{2}(1 - r_1) > 0$. Hence $\Delta R_1 > 0$. In much the same way $\Delta E_1 > 0$ can be proved.

Assume that we have already constructed sets $E_1, \dots, E_n$ in H and positive numbers $r_1, \dots, r_n$ with the following properties:

(a) $E_k = R_k \cup E_{k-1}$, where $R_k = E_{k-1} + r_k E$, $1 \leq k \leq n$;

(b) $r_k < 1/k$ and $\Delta E_k > 0$ for $1 \leq k \leq n$.

Fix any positive number $r_{n+1} < \min\big(\frac{1}{n+1}, \frac{\Delta E_n}{\sqrt{2}}\big)$ and let $E_{n+1} = R_{n+1} \cup E_n$, $R_{n+1} = E_n + r_{n+1}E$. It is easily checked that $\Delta E_{n+1} \geq \min\big(r_{n+1}, \Delta E_n - r_{n+1}\sqrt{2}\,\big) > 0$.

Thus, we have inductively constructed a sequence of sets $\{E_n : n \in \mathbb{N}\}$, $E_n \subset H$, and a sequence of positive numbers $\{r_n : n \in \mathbb{N}\}$, $r_n < 1/n$, such that

$$E_{n+1} = E_n \cup (E_n + r_n E), \quad \Delta E_n > 0. \tag{9}$$

Put $X_0 = \bigcup_{n=1}^{\infty} E_n$. Since $\Delta E_n > 0$ and $|E_n| = |E| = \kappa$ we have that X_0 is σ-discrete and $|X_0| = \kappa$. Let $B(x,r)$ be a fixed open ball in X_0. There is n such that $x \in E_n$. Then from the recurrent relation (9) it follows that $x + r_{n+p}E \subset B(x,r)$ whenever $r_{n+p} < r/\sqrt{2}$ (notice that $\operatorname{diam} E = \sqrt{2}$). Since $r_m \to 0$, we conclude that for all sufficiently large p

$$\kappa = |E| = \left|x + r_{n+p}E\right| \le \left|B(x,r)\right| \le |X_0| = \kappa .$$

Thus each ball in X_0 has cardinality κ whence, in particular, X_0 is dense in itself.

Finally, let $(\overline{H}, \varphi)$ be a completion of H where $\varphi \colon H \to \overline{H}$ is an isometric imbedding. Then $X = \varphi(X_0)$ is a metric subspace of the Hilbert space $\overline{H}$ having the required properties. □

4 On $\mathcal{F}_\sigma$-sets and the relation $w(X) = |X|$

By GCH and MA we mean the Generalized Continuum Hypothesis and Martin's Axiom, respectively. In what follows, we will need some facts which are easy consequences from the previous results on σ-discrete spaces. It will be convenient to formulate them as

Theorem 7. *For each σ-discrete metric space X we have*

(a) $w(X) = |X|$;

(b) *each $E \subset X$ is $\mathcal{F}_\sigma$;*

(c) *each $f \colon X \to \mathbb{R}$ is Baire 1.*

The relation (a) was established in Theorem 3, (b) follows immediately from Lemma 1 while (c) is equivalent to (b).

In this connection we are interested in the following questions. Do these relations imply that X is σ-discrete? And, how can be compared (a) and (b)?

Theorem 8. *Assume GCH. If each subset of an infinite metric space X is $\mathcal{F}_\sigma$ then $w(X) = |X|$.*

P r o o f. It suffices to show that

$$w(X) \ge |X| \tag{10}$$

since the opposite inequality always holds for metric spaces. We use the notations:

$o(X) =$ cardinality of the family of all open subsets of X.

$f(X) =$ cardinality of the family of all closed subsets of X.

$f_\sigma(X) =$ cardinality of the family of all $\mathcal{F}_\sigma$-subsets of X.

It is known that $o(X) = 2^{w(X)}$ ([6], p. 33), therefore $f(X) = 2^{w(X)}$. It follows that $f_\sigma(X)$ has the bound $(2^{w(X)})^{\aleph_0} = 2^{w(X)}$ because $w(X) \geq \aleph_0$.

Now assume (10) is false. Then using the bound for $f_\sigma(X)$ and the GCH assumption, we get $f_\sigma(X) \leq 2^{w(X)} < 2^{|X|}$. Therefore there is a set $A \subset X$ which is not $\mathcal{F}_\sigma$, a contradiction. So (10) holds. □

Without any assumption whatsoever about X, the statement of Theorem 8 may fail. To see this, let us look at the so-called Q-sets.

Recall that an uncountable set $E \subset \mathbb{R}$ is called a *Q-set* if each subset of E is $\mathcal{F}_\sigma$ in the space E (equipped with the subspace topology). It is known (see [6], p. 709) that under the assumption $MA + \neg CH$, each $E \subset \mathbb{R}$ satisfying $\aleph_0 < |E| < \mathfrak{c}$ is a Q-set (cf. also [7], [8]).

Each Q-set E, as a subspace of $\mathbb{R}$, is obviously separable and not σ-discrete. It follows by Theorem 2 that there exist massive Q-sets. Indeed, given a Q-set E its subset $\mathcal{M}(E)$ is massive. It will be convenient to sum up the mentioned facts in the form of the following proposition.

Theorem 9. *Under $MA + \neg CH$, there exists an uncountable metric space $X \subset \mathbb{R}$ (namely, a Q-set) such that*

1) X is massive;

2) $w(X) = \aleph_0 < |X|$;

3) each subset of X is $\mathcal{F}_\sigma$.

Thus the relation (b) in Theorem 7 generally need not imply (a). Consider the space X in Example 2. It is easy to check out that $w(X) = |X| = \mathfrak{c}$ but the set of irrationals in the interval $[0,1]_y$, $y \in \mathbb{R}$, is not $\mathcal{F}_\sigma$. The same property could be shown in a similar way for the “hedgehog” $J(\mathfrak{c})$. Hence the relation (a) need not imply (b) either. Thus the relations (a), (b) are generally independent.

Next, we will show that the converse of Theorem 7 is not true, i.e., (a)+(b) need not imply that a metric space X is σ-discrete.

Example 3. Assume $MA + \neg CH$ and let $E \subset [0,1]$ be a massive Q-set. Consider the metric space quite analogous to that in Example 2 where $\mathfrak{c}$ copies of $[0,1]$ are replaced with $|E|$ copies E_y of E:

$$X = \bigcup_{y \in E} E_y .$$

It is clear that X is a massive space.

First, we check that $w(X) = |X| = |E|$. Let $\mathcal{B} = \{b_t : t \in T\}$ be a base for X such that $w(X) = |\mathcal{B}|$ and let $\mathcal{B}_y = \{b_t : b_t \subset [0,1]_y\}$, $y \in E$. Since the family $\{E_y : y \in E\}$ is obviously discrete (cf. [5]) we have that $\mathcal{B}_y$ is a base for E_y, $y \in E$. It follows that $\mathcal{B}^* = \bigcup_{y \in E} \mathcal{B}_y \subset \mathcal{B}$ is a base for X, and as $\mathfrak{a} \le |\mathcal{B}_y| \le |E|$, we get

$$w(X) = |\mathcal{B}^*| = \sum_{y \in E} |B_y| = |E| = |X|.$$

We claim that each subset of X is $\mathcal{F}_\sigma$. Let $A \subset X$. Then $A = \bigcup_{y \in E} A \cap E_y$. Since E_y is a Q-set, each $A \cap E_y$ is $\mathcal{F}_\sigma$ in the closed subset E_y of X. Our claim then follows from the fact that the family $\{E_y : y \in E\}$ is discrete. But X is obviously not σ-discrete. Thus generally the converse of Theorem 7 does not hold.

However, the converse of Theorem 7 holds under some additional assumptions (see Theorem 10).

Definition 6. A topological space X is called a *Q-space* if each of its subsets is $\mathcal{F}_\sigma$.

For example, σ-discrete metric spaces and Q-sets are Q-spaces.

Theorem 10. *Let X be a complete metric space such that $\Sigma(X) \neq \emptyset$. If X is a Q-space then X is σ-discrete, i.e., $X = \Sigma(X)$.*

Theorem 10 will be given two proofs. The first one is based on the following Lemma 3. In the second proof we will make use of Theorem 12 on the existence of ω-primitives.

Lemma 3. *Let X be a dense in itself metric space.*

(i) *Then X contains a dense subset of the form*

$$D = \bigcup_{k=1}^{\infty} D_k \tag{11}$$

where $\Delta D_k > 0$ and D_k are pairwise disjoint. Thus D is $\mathcal{F}_\sigma$, first category and σ-discrete.

(ii) *If X is a massive space or a dense in itself Baire space, then D is boundary.*

P r o o f . Although formally the proof can be reconstructed from certain arguments of Theorem 1 and Lemma 3 in [1], and also Theorem 1.2 in [4], we will give, for the sake of reader's convenience, a comprehensive proof of the lemma, especially because it was not stated at all in such a form in the mentioned papers.

For each natural n consider the property P_n pertained to subsets of X and defined as follows

$$E \subset X \text{ has the property } P_n \iff \Delta E \geq 1/n\,.$$

It is easily checked that P_n is a property of finite character [5].

According to Teichmüller–Tukey's Lemma [5], each subset of X with property P_n is contained in the maximal (with respect to the inclusion relation) subset of X which also has the property P_n. Such maximal subset we call a P_n-maximal set (for more details see [1]). So, if we fix, say, a one-element subset of X, we may conclude that it is contained in a P_1-maximal set $A_1 \subset X$.

Now, let us note that since X is dense in itself, then for each $A \subset X$ with $\Delta A > 0$ we have that $X \setminus A$ is also dense in itself. Indeed, A is closed and discrete whence the claim follows. So, in particular, $X \setminus A_1$ is dense in itself. Proceed inductively. Assume that we have already found nonempty subsets $D_1, \dots, D_n$ of X such that

(a) $D_1 \subset X$, $D_k \subset X \setminus (D_1 \cup \dots \cup D_{k-1})$ for $k = 2, \dots, n$,

(b) $\Delta D_k > 1/k$, $1 \leq k \leq n$,

(c) D_1 is P_1-maximal in X, and D_k is P_k-maximal in $X \setminus (D_1 \cup \dots \cup D_{k-1})$ for $k = 2, \dots, n$.

Repeating much the same argument as for the set $X \setminus A$ above, one easily concludes that $X \setminus (D_1 \cup \dots \cup D_n)$ is dense in itself. We may then find (by Teichmüller-Tukey's lemma) a P_{n+1}-maximal set $D_{n+1} \subset X \setminus (D_1 \cup \dots \cup D_n)$.

Thus there exists a sequence $\{D_k,\ k \in \mathbb{N}\}$ of subsets of X satisfying the properties (a), (b), (c) for all k. We claim that

$$D = \bigcup_{k=1}^{\infty} D_k$$

is the required set.

Let us show that D is dense in X. Suppose not. Then there is a ball $B(x_0, r) \subset X \setminus D$. It follows that $\Delta\big(\{x_0\} \cup D_k\big) = \Delta D_k > 1/k$ for $k > 1/r$. This means that D_k is not a P_k-maximal set (cf. the property (c)), a contradiction. Thus the set D is dense in X and, in view of its construction, it has the properties indicated in (i) of the statement of the lemma. The claim (ii) is an easy consequence of these properties. □

F i r s t p r o o f o f T h e o r e m 10. By Theorem 2, $X = \Sigma(X) \cup \mathcal{M}(X)$, so it remains to show that $\mathcal{M}(X) = \emptyset$. Assume this is not the case. Then the massive subspace $\mathcal{M}(X)$ contains, by virtue of Lemma 3, a dense first category boundary subset D. By assumption, X is a Q-space, therefore $\mathcal{M}(X) \setminus D$ should be an $\mathcal{F}_\sigma$ subset of a complete space $\mathcal{M}(X)$. Thus we have a contradiction because a complete space cannot be a disjoint union of a first category dense set and of an $\mathcal{F}_\sigma$ set. □

THEOREM 11. *Let X be a Baire metric Q-space. Then $X = \overline{I(X)}$ where $I(X)$ is the set of isolated points of X (and $\overline{I(X)}$ means the closure of $I(X)$).*

P r o o f . Assume the contrary. Then the nonempty open set $G = X \setminus \overline{I(X)}$ is a dense in itself Baire space. By Lemma 3, G contains a dense first category boundary set D. As X is a Q-space, the set $G \setminus D$ should be $\mathcal{F}_\sigma$ which is obviously impossible since G is Baire (cf. end of the proof of Theorem 10). ☐

COROLLARY. *There are no dense in themselves Baire metric Q-spaces.*

Next, let us give an alternative proof of Theorem 10 making use of the following fact.

THEOREM 12. *Let X be a dense in itself metric space. Then each* USC *function $f\colon X \to [0, \infty)$ has an ω-primitive $F\colon X \to \mathbb{R}$.*

This is a particular case of a fundamental Theorem 3 in [3] on the existence of ω-primitives on any metric space, not necessarily dense in itself.

S e c o n d p r o o f o f T h e o r e m 10. By Theorem 2, $X = \Sigma(X) \cup \mathcal{M}(X)$. Assuming that $\mathcal{M}(X) \neq \emptyset$, consider the constant function $f\colon \mathcal{M}(X) \to \{1\}$ which is obviously USC. By Theorem 12, there is an ω-primitive $F\colon \mathcal{M}(X) \to \mathbb{R}$ for f which is Baire 1 because X is a Q-space. Since X is complete, the closed subspace $\mathcal{M}(X)$ is also complete. Therefore the oscillation of F vanishes at some point, which contradicts the fact that $\omega(F, \cdot) = 1$. Thus $\mathcal{M}(X) = \emptyset$ whence $X = \Sigma(X)$, and this finishes the proof. ☐

Theorem 11 can be handled in a similar way.

S e c o n d p r o o f o f T h e o r e m 11. Retaining the notations from the first proof, assume that $G \neq \emptyset$. By Theorem 12, there is an ω-primitive $F\colon G \to \mathbb{R}$ for the constant function $f\colon G \to \{1\}$. Since X is a Q-set, F is Baire 1, therefore its oscillation vanishes at some point of the Baire space G, a contradiction. ☐

Remark 3. As we have already noted in the proofs of Theorem 10, if X is complete then the subspace $\mathcal{M}(X)$ is also complete, whenever nonempty. In Example 4 it is shown that a similar "hereditary property" as well as the analogue of Theorem 10 fails if "complete space" is replaced with "Baire metric space". The example is partially based on Theorem 13.

THEOREM 13. *Let X be a Q-space. Then for each σ-discrete metric space Y the product $Z = X \times Y$ is a Q-space.*

P r o o f . Assume first that Y is a discrete space. Write Y as a union of singletons: $Y = \bigcup_{t \in T} \{y_t\}$. Let $A \subset Z$. For each $t \in T$ there is a set $E_t \subset X$ such that $A \cap (X \times \{y_t\}) = E_t \times \{y_t\}$. By our assumption, $E_t = \bigcup_{k=1}^{\infty} F_{tk}$ where F_{tk} are closed sets in the space X. Then we have

$$A = \bigcup_{t \in T} E_t \times \{y_t\} = \bigcup_{k=1}^{\infty} \left(\bigcup_{t \in T} F_{tk} \times \{y_t\} \right). \tag{12}$$

Each set $F_{tk} \times \{y_t\}$ is closed in Z, and since Y is discrete, the family $\{F_{tk} \times \{y_t\} \colon t \in T\}$ is discrete (cf. [5]) for each $k \in \mathbb{N}$. Therefore the set $\bigcup_{t \in T} F_{tk} \times \{y_t\}$ is closed in Z for each $k \in \mathbb{N}$. It follows by (12) that A is $\mathcal{F}_\sigma$.

Now let Y be a σ-discrete metric space. By Lemma 1, $Y = \bigcup_{n=1}^{\infty} Y_n$ where the sets Y_n are discrete and closed. Let $A \subset X \times Y$. Write $A = \bigcup_{n=1}^{\infty} A \cap (X \times Y_n)$. By the previous argument, each set $A \cap (X \times Y_n)$ is $\mathcal{F}_\sigma$ in the closed subspace $X \times Y_n$ of the space Z. Therefore the set A is $\mathcal{F}_\sigma$. □

Remark 4. The following two questions seem to be interesting. Let X be a Q-space. For which spaces Y the product $X \times Y$ is a Q-space (or, conversely, is not a Q-space)? In this connection one should note that the product of two Q-spaces need not be a Q-space. This is immediate from a result of W. G. Fleissner in [9]. Second question. Assume that a given space Y is not σ-discrete. Is there a Q-space X such that $X \times Y$ is a Q-space?

Finally, let us consider an example explaining what was said in Remark 3.

Example 4. Assume $MA + \neg CH$. Let a massive Q-set $E \subset [0,1]$ be fixed and let $A = \{a_1, \dots, a_n, \dots\} \subset E$ be a countable set dense in E. Consider the discrete set $Y = \{1, 1/2, \dots, 1/n, \dots\} \subset \mathbb{R}$. By Theorem 13, the product $E \times Y$ is a Q-space. Define its subspace X letting

$$X = ((E \times \{0\}) \cup D, \quad \text{where} \quad D = \{(a_1, 1), \dots, (a_n, 1/n), \dots\}. \tag{13}$$

Clearly, X is a Q-space and D is its discrete subset. We shall show that X is a Baire space. To end this, it suffices to check that D is dense in X, which reduces to showing that $E \times \{0\}$ is contained in the closure $\overline{D}$.

Take any open ball $B(p,r) \subset \mathbb{R}^2$ around a point $p = (x,0) \in E \times \{0\}$. Since A is dense in a massive space E, there is a subsequence $\{a_{n_k} : k \in \mathbb{N}\}$, $a_{n_k} \neq x$, $a_{n_k} \to x$. By (13), this implies that $D \cap B(p,r) \neq \emptyset$, whence $p \in \overline{D}$. Thus $X = \overline{D}$, and since D is discrete, we conclude that X is a (metrizable) Baire space (for concreteness, we may consider the standard euclidean metric from $\mathbb{R}^2$) on X.

Now let us observe that $\Sigma(X) = D$, $\mathcal{M}(X) = E \times \{0\}$. The massive space $\mathcal{M}(X)$ homeomorphic to the space E is not Baire since E is such. In fact, E is first category in itself (cf. [8]).

Thus Example 4 shows that (under $MA + \neg CH$) there exists a Baire metric Q-space X such that $\Sigma(X) \neq \emptyset$, $\mathcal{M}(X) \neq \emptyset$, and $\mathcal{M}(X)$ is not a Baire space.

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Received October 16, 2002

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