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ON MEASURABLE SPACES AND MEASURABLE MAPS

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ABSTRACT. We introduce and study the category ID the objects of which are suitable convergence D-posets of maps into the closed unit interval I and the morphisms of which are sequentially continuous D-homomorphisms. We show that ID is dual to a subcategory of the category MID of generalized measurable spaces and generalized measurable maps. We construct epireflective and monocoreflective subcategories of ID and MID corresponding to two important properties of objects in ID, soberness and sequential closedness. The subcategories play important roles in applications to probability. We generalize some basic probability notions so that the generalized random variables are dual to generalized observables and generalized probability measures are ID-morphisms. Let I_u be an ultrapower of I. We modify some of the previous results replacing I by I_u and replacing the sequential convergence by the approximation: a sequence approximates a point in I_u whenever the sequence of standard parts converges to the standard part of the point in I.

0. Introduction

Let $\mathbb{A}$ be a field of subsets of X, let $\mathbb{B}$ be a field of subsets of Y, and let f be a map of Y into X. Then f is said to be $(\mathbb{A},\mathbb{B})$-measurable if for each $A \in \mathbb{A}$ the preimage $f^{\leftarrow}(A) = \{y \in Y : f(y) \in A\}$ belongs to $\mathbb{B}$. It can be readily seen that f is $(\mathbb{A},\mathbb{B})$-measurable iff for each characteristic function $\chi_A : X \to \{0,1\}$, $A \in \mathbb{A}$, the composition $\chi_A \circ f$ is a characteristic function of some $B \in \mathbb{B}$. This leads to various generalizations of the measurability (cf. Frič [14]). The categorical theory (cf. Herrlich and Strecker [18]) and in particular the language of mathematical structure (cf. Adámek [1, 2]) is

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natural and suitable to carry out and describe various constructions used in the foundations of generalized probability, i.e., the duality between observables and random variables. The sequential convergence and sequential continuity of the morphisms plays a key role (cf. Frič [11, 13, 14]). Even though some of our results are stated in categorical terms, we provide simple and elementary proofs. This way probabilistic interpretation is more natural and leads to natural generalizations. We mention the operational random variables (cf. Bugajski [3, 4], Gudder [17]) and propose a generalization using the D-posets (cf. Kôpka and Chovanec [22] generalization using the D-posets). In the last section we modify some constructions of the previous sections by replacing I by its nonstandard counterpart I_u.

MV-algebras (cf. Cignoli, D'ottaviano and Mundici [6]) generalize Boolean algebras and many fundamental constructions and theorems of the classical Kolmogorovian probability theory have been generalized to MV-algebras and, in particular, to MV-algebras with product (cf. Dvurečenskij and Pulmannová [8], Mundici and Riečan [23], Jurečková [21], Petrovičová [26, 27]). Therefore, it might be interesting to find out whether similar results can be extended to a suitable subcategory of D-posets with an additional product operation.

1. D(I)-posets

In Frič [14] it is suggested to study the subcategory of D-posets cogenerated by the closed unit interval in connection with the generalized probability. In this section we give basic definitions and describe basic properties of this subcategory. Our constructions are rather natural and explicit, the proofs are direct and elementary. We show that our categories ID, MID and their distinguished subcategories of sober objects fit the general duality pattern described in Frič [17].

Recall (cf. Kôpka and Chovanec [22], Chovanec and Kôpka [5]) that a *D-poset* is a quintuple $(X, \leq, \ominus, 0_X, 1_X)$ where X is a set, $\leq$ is a partial order, 0_X is the least element, 1_X is the greatest element, $\ominus$ is partial operation on X such that $a \ominus b$ is defined iff $b \leq a$, and the following axioms are assumed:

(D1) $a \ominus 0_X = a$ for each $a \in X$;

(D2) If $c \leq b \leq a$, then $a \ominus b \leq a \ominus c$ and $(a \ominus c) \ominus (a \ominus b) = b \ominus c$.

If no confusion can arise, then the quintuple $(X, \leq, \ominus, 0_X, 1_X)$ is condensed to X. A map h of a D-poset X into a D-poset Y which preserves the D-structure is said to be a *D-homomorphism*.

It is known that D-posets are equivalent to the effect algebras introduced in Foulis and Bennett [9]. Interesting results about effect algebras and

D-posets can be found in Dvurečenskij and Pulmannová [8] and in a series of papers by Z. Riečanová (cf. Riečanová [29]) and G. Jenča (cf. Jenča [19]).

Denote I the closed unit interval carrying the usual linear order and the usual D-structure: $a \ominus b$ is defined whenever $b \le a$ and then $a \ominus b = a - b$. Analogously, if X is a set and I^X is the set of all functions on X into I, then we consider I^X as a D-poset in which the partial order and the partial operation $\ominus$ are defined pointwise: $b \le a$ iff $b(x) \le a(x)$ for all $x \in X$ and $a \ominus b$ is defined by $(a \ominus b)(x) = a(x) - b(x)$, $x \in X$. A subset $\mathcal{X} \subseteq I^X$ containing the constant functions $0_{\mathcal{X}}$, $1_{\mathcal{X}}$ and closed with respect to the inherited partial operation "$\ominus$" is a typical D-poset we are interested in.

There is an extensive list of papers devoted to sequential convergence on various algebraic structures. Interesting results about sequential convergence on MV-algebras can be found in Jakubík [20].

By a sequential convergence on a set X we understand a subset $\mathbb{L} \subseteq X^{\mathbb{N}} \times X$ satisfying the following conditions:

(S) If $\langle x_n \rangle$ is a constant sequence and $x_n = x$, $n \in \mathbb{N}$, then $(\langle x_n \rangle, x) \in \mathbb{L}$;

(F) If $(\langle x_n \rangle, x) \in \mathbb{L}$ and $\langle x'_n \rangle$ is a subsequence of $\langle x_n \rangle$, then $(\langle x'_n \rangle, x) \in \mathbb{L}$.

Definition 1.1. Let $(X, \le, \ominus, 0_X, 1)$ be a D-poset. Let $\mathbb{L} \subseteq X^{\mathbb{N}} \times X$ be a sequential convergence on X such that

(D) If $(\langle x_n \rangle, x) \in \mathbb{L}$, $(\langle y_n \rangle, y) \in \mathbb{L}$ and $y_n \le x_n$, $n \in \mathbb{N}$, then $y \le x$ and $(\langle x_n \ominus y_n \rangle, x \ominus y) \in \mathbb{L}$.

Then $(X, \le, \ominus, 0_X, 1_X, \mathbb{L})$ is said to be a *convergence D-poset*.

In what follows, we shall always assume that the sequential convergence $\mathbb{L}$ in a convergence D-poset X satisfies the following additional conditions:

(H) If $(\langle x_n \rangle, x) \in \mathbb{L}$ and $(\langle x_n \rangle, y) \in \mathbb{L}$, then $x = y$ (the uniqueness of limits);

(U) If $\langle x_n \rangle$ is a sequence and $x \in X$ is a point such that for each subsequence $\langle x'_n \rangle$ of $\langle x_n \rangle$ there exists a subsequence $\langle x''_n \rangle$ of $\langle x'_n \rangle$ such that $(\langle x''_n \rangle, x) \in \mathbb{L}$, then $(\langle x_n \rangle, x) \in \mathbb{L}$ (Fréchet–Urysohn condition);

(B) If $(\langle a_n \rangle, x)$, $(\langle b_n \rangle, x) \in \mathbb{L}$ and $a_n \le x_n \le b_n$, $n \in \mathbb{N}$, then $(\langle x_n \rangle, x) \in \mathbb{L}$.

Even though in probability theory the monotone convergence and the sequential continuity with respect to the monotone convergence are standard (cf. Riečan and Neubrunn [28], Dvurečenskij and Pulmannová [8]), in accordance with Frič [10, 11, 12, 13, 14, 15], we shall work with the usual convergence in I and the pointwise (the product convergence) in I^X. Observe that, for the usual probability domains like fields sets or bold algebras, if an additive generalized probability measure is sequentially continuous with respect to the monotone convergence, then it is sequentially continuous with respect to the pointwise convergence, as well (cf. F r i č [15]).

Let X be a set and let $\mathcal{X}$ be a set of maps on X into I such that $\mathcal{X}$ is a D-poset with respect to the pointwise partial order and the pointwise difference operation "$\ominus$" inherited from I^X, i.e., $0_{\mathcal{X}}, 1_{\mathcal{X}} \in \mathcal{X}$ and if $a, b \in \mathcal{X}$, $b(x) \leq a(x)$ for all $x \in X$, then $a \ominus b \in \mathcal{X}$. The proof of the next lemma is trivial and it is omitted.

LEMMA 1.2. *Let $\mathbb{L}$ be the pointwise convergence on $\mathcal{X}$. Then $(\mathcal{X}, \leq, \ominus, 0_{\mathcal{X}}, 1_{\mathcal{X}}, \mathbb{L})$ is a convergence* D*-poset and $\mathbb{L}$ satisfies the additional conditions* (H), (U), (B).

DEFINITION 1.3. The sixtuple $(\mathcal{X}, \leq, \ominus, 0_{\mathcal{X}}, 1_{\mathcal{X}}, \mathbb{L})$ is said to be a $\mathrm{D}(I)$-*poset*.

Denote $\mathrm{D}(I)$ the category the objects of which are $\mathrm{D}(I)$-posets and the morphisms of which are sequentially continuous D-homomorphisms. Let $(\mathcal{X}, \leq, \ominus, 0_{\mathcal{X}}, 1_{\mathcal{X}}, \mathbb{L})$ be a $\mathrm{D}(I)$-poset. If no confusion can arise, we shall say that $\mathcal{X} \subseteq I^X$, or simply $\mathcal{X}$, is a $\mathrm{D}(I)$-poset. Each $x \in X$ can be identified with a sequentially continuous D-homomorphism, called the evaluation of $\mathcal{X}$ at $x \in X$, $\mathrm{ev}_x \colon \mathcal{X} \to I$ defined by $\mathrm{ev}_x(a) = a(x)$, $a \in \mathcal{X}$.

DEFINITION 1.4. If for each morphism $h \colon \mathcal{X} \to I$ there exists $x \in X$ such that $h = \mathrm{ev}_x$, then $\mathcal{X}$ is said to be *sober*. If for each $x, y \in X$, $x \neq y$, there exists $a \in \mathcal{X}$ such that $a(x) \neq a(y)$, then $\mathcal{X}$ is said to be *reduced*.

Observe that if $\mathcal{X}$ is sober and reduced, than for each morphism $h \colon \mathcal{X} \to I$ there exists exactly one $x \in X$ such that $h = \mathrm{ev}_x$.

Consider the equivalence relation on X defined by $x \sim y$ iff $a(x) = a(y)$ for all $a \in \mathcal{X}$, choose from each equivalence class exactly one point and denote $X^{\sim}$ the resulting subset of X, denote $\mathcal{X}^{\sim}$ the set $\mathcal{X} \restriction X^{\sim} \subseteq I^{X^{\sim}}$ of all restrictions $a \restriction X^{\sim}$, $a \in \mathcal{X}$.

LEMMA 1.5. *The* $\mathrm{D}(I)$*-poset $\mathcal{X}^{\sim}$ is reduced and $\mathcal{X}$ and $\mathcal{X}^{\sim}$ are isomorphic.*

P r o o f . The proof is trivial and it is omitted. $\square$

In view of Lemma 1.5, we shall focus on reduced $\mathrm{D}(I)$-posets.

Let $\mathcal{X} \subseteq I^X$ be a reduced $\mathrm{D}(I)$-poset. Denote X^* the set of all morphisms of $\mathcal{X}$ into I and, identifying each $x \in X$ with the morphism $\mathrm{ev}_x \colon \mathcal{X} \to I$, consider X as a subset of X^*. For each $a \in \mathcal{X}$, define a mapping a^* of X^* into I as follows: $a^*(x) = x(a)$. Trivially, $a^* = b^*$ iff $a = b$. Clearly, this yields a $\mathrm{D}(I)$-poset $\mathcal{X}^* \subseteq I^{X^*}$ such that $\mathcal{X}^* \restriction X = \mathcal{X}$. Finally, define a map e of $\mathcal{X}$ into $\mathcal{X}^*$ by putting $e(a) = a^*$, $a \in \mathcal{X}$. The straightforward proof of the next lemma is omitted.

Lemma 1.6. (i) *The map e is an isomorphism of $\mathcal{X}$ onto $\mathcal{X}^*$.*

(ii) *The* $\mathrm{D}(I)$*-poset $\mathcal{X}^* \subseteq I^{X^*}$ is sober and reduced.*

(iii) *Let h be a sequentially continuous* D*-homomorphism of $\mathcal{X}$ into a sober* $\mathrm{D}(I)$*-poset $\mathcal{Y} \subseteq I^Y$. Define a map h^* of $\mathcal{X}^*$ into $\mathcal{Y}$ as follows: $h^*(a^*) = h(a)$. Then h^* is a sequentially continuous* D*-homomorphism of $\mathcal{X}^*$ into $\mathcal{Y}$.*

Definition 1.7. Let $\mathcal{X} \subseteq I^X$ be a $\mathrm{D}(I)$-poset. Then $(X, \mathcal{X})$ is said to be a $\mathrm{D}(I)$-*measurable space*, or simply a *measurable space*. If $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ are $\mathrm{D}(I)$-measurable spaces and f is a map of Y into X such that $a \circ f \in \mathcal{Y}$ for all $a \in \mathcal{X}$, i.e., $\mathcal{X} \circ f \subseteq \mathcal{Y}$, then f is said to be $(\mathcal{X}, \mathcal{Y})$-*measurable*, or simply *measurable*. If $\mathcal{X}$ is sober or reduced, then $(X, \mathcal{X})$ is said to be *sober* or *reduced*.

Let $\mathrm{MD}(I)$ be the category the objects of which are $\mathrm{D}(I)$-measurable spaces and the morphisms of which are measurable maps.

Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be ID-measurable spaces and let f be an $(\mathcal{X}, \mathcal{Y})$-measurable map. Then $a \circ f$, $a \in \mathcal{X}$, defines a map of $\mathcal{X}$ into $\mathcal{Y}$; denote it $f^{\triangleleft}$.

Lemma 1.8. *$f^{\triangleleft}$ is a sequentially continuous* D*-homomorphism of the* $\mathrm{D}(I)$*--poset $\mathcal{X}$ into the* $\mathrm{D}(I)$*-poset $\mathcal{Y}$.*

Proof. Clearly, $0_{\mathcal{X}} \circ f = 0_{\mathcal{Y}}$ and $1_{\mathcal{X}} \circ f = 1_{\mathcal{Y}}$. Let $a, b \in \mathcal{X}$, $b \le a$. Then $b(x) \le a(x)$ for all $x \in X$ and hence $\big(f^{\triangleleft}(b)\big)(y) = b\big(f(y)\big) \le a\big(f(y)\big) = \big(f^{\triangleleft}(a)\big)(y)$ for all $y \in Y$. Thus $f^{\triangleleft}(b) \le f^{\triangleleft}(a)$. Further, $\big(f^{\triangleleft}(a \ominus b)\big)(y) = (a \ominus b)\big(f(y)\big) = a\big(f(y)\big) \ominus b\big(f(y)\big) = \big(f^{\triangleleft}(a)\big)(y) \ominus \big(f^{\triangleleft}(a)\big)(y)$ for all $y \in Y$ and hence $f^{\triangleleft}$ is a D-homomorphism. Finally, assume that a sequence $\langle a_n \rangle$ converges in $\mathcal{X}$ to a. Then each sequence $\langle a_n(x) \rangle$ converges to $a(x)$, $x \in X$. But $a_n\big(f(y)\big) = (a_n \circ f)(y) = \big(f^{\triangleleft}(a_n)\big)(y)$ implies that, for each $y \in Y$, the sequence $\langle \big(f^{\triangleleft}(a_n)\big)(y) \rangle$ converges to $\big(f^{\triangleleft}(a)\big)(y)$. Consequently, $\langle f^{\triangleleft}(a_n) \rangle$ converges in $\mathcal{Y}$ to $f^{\triangleleft}(a)$ and $f^{\triangleleft}$ is sequentially continuous. □

Lemma 1.9. *Let f_1, f_2 be $(\mathcal{X}, \mathcal{Y})$-measurable maps and let $f_1^{\triangleleft}$, $f_2^{\triangleleft}$ be the corresponding sequentially continuous* D*-homomorphisms of $\mathcal{X}$ into $\mathcal{Y}$. If $\mathcal{X}$ is reduced, then $f_1 \ne f_2$ implies $f_1^{\triangleleft} \ne f_2^{\triangleleft}$.*

Proof. The proof is straightforward and it is omitted. □

Lemma 1.10. *Let $\mathcal{X}$ be reduced and sober and let h be a sequentially continuous* D*-homomorphism of $\mathcal{X}$ into $\mathcal{Y}$. Then there exists a unique $(\mathcal{X}, \mathcal{Y})$-measurable map g such that $h = g^{\triangleleft}$.*

Proof. For each $y \in Y$, the composition $\mathrm{ev}_y \circ h$ is a sequentially continuous D-homomorphism of $\mathcal{X}$ into I. Since $\mathcal{X}$ is reduced and sober, there exists a unique $x \in X$ such that $\mathrm{ev}_x = \mathrm{ev}_y \circ h$. Put $x = g(y)$. This defines a map g

of Y into X. A straightforward calculation shows that $h = g^{\triangleleft}$. It follows from Lemma 1.9 that g is determined uniquely. □

Observe that if $\mathcal{X} \subseteq I^X$ fails to be sober, $\mathcal{X}^* \subseteq I^{X^*}$ is the D(I)-poset from Lemma 1.6, and $e\colon \mathcal{X} \to \mathcal{X}^*$ is the isomorphism sending a into a^*, then there is no $(\mathcal{X}, \mathcal{X}^*)$-measurable map g of X^* into X such that $e = g^{\triangleleft}$. Hence the soberness of $\mathcal{X}$ in Lemma 1.6 cannot be omitted.

In what follows, a blanket assumption is made that *all* D(I)*-posets are reduced.* Let ID be the (full) subcategory of D(I) consisting of reduced D(I)-posets and, accordingly, a reduced D(I)-poset will be called an ID-*object.* Similarly, a reduced D(I)-measurable space will be called an ID-*measurable space* and MID will be the (full) subcategory of MD(I) consisting of reduced D(I)-measurable spaces.

Let SID be the (full) subcategory of ID consisting of reduced sober D(I)-posets and let MSID be the (full) subcategory of MID consisting of reduced sober ID-measurable spaces. In view of Lemma 1.8, Lemma 1.9, Lemma 1.10, passing from $\mathcal{X} \subseteq I^X$ to $(X, \mathcal{X})$ yields a contravariant functor from ID to MID and, vice versa, passing from $(X, \mathcal{X})$ to $\mathcal{X} \subseteq I^X$ yields a contravariant functor from MSID to SID.

Theorem 1.11. *The categories* SID *and* MSID *are dually isomorphic.*

P r o o f . It follows from Lemma 1.8, Lemma 1.9, Lemma 1.10 that the compositions of the two contravariant functors result in the identity functors on SID and MSID, respectively. □

Theorem 1.12. (i) *The categories* ID *and* SID *are naturally equivalent.*

(ii) *The categories* ID *and* MSID *are dually equivalent.*

P r o o f . (i) follows from Lemma 1.6. (ii) follows from (i) and Theorem 1.11. □

Consider the category D of D-posets and D-homomorphisms and consider its distinguished object I. Let D_I be the concrete category defined as follows. Since D has products, each power I^X and its reduced subsets $\mathcal{X} \subseteq I^X$, closed with respect to the inherited D-poset structure and carrying the pointwise sequential convergence, form the objects of D_I and sequentially continuous D-homomorphisms form the morphisms of D_I. Observe that, if fact, D_I coincides with our category ID. Via a general categorical construction, it is shown in F r i č [14] that to D_I there corresponds a category MD_I of generalized measurable spaces and distinguished subcategory of MD_I is dually equivalent to D_I. This way the duality described in this section becomes a special case of the duality constructed in F r i č [14].

2. Extension

In this section we construct two types of extensions of an ID-object $\mathcal{X} \subseteq I^X$, one extending the domain set X and the other one adding more functions to $\mathcal{X}$.

Definition 2.1. Let $(X,\mathcal{X})$ and $(Y,\mathcal{Y})$ be ID-measurable spaces. Let X be a subset of Y and let $\mathcal{X}$ be the restriction $\mathcal{Y} \restriction X$ of $\mathcal{Y}$ to X. Then $(X,\mathcal{X})$ is said to be a subspace of $(Y,\mathcal{Y})$. If $a = b$ whenever $a, b \in \mathcal{Y}$ and $a \restriction X = b \restriction X$, then $(X,\mathcal{X})$ is said to be a *dense* subspace.

Let $(X,\mathcal{X})$ be an ID-measurable space and let $(X^*,\mathcal{X}^*)$ be the sober ID-measurable space described in Lemma 1.6. If X is sober, then $X^* = X$ and $\mathcal{X}^* = \mathcal{X}$.

Lemma 2.2. (i) *$(X,\mathcal{X})$ is a dense subspace of $(X^*,\mathcal{X}^*)$.*

(ii) *Let f be a measurable map of $(X,\mathcal{X})$ into a sober* ID-*measurable* ID-*space $(Y,\mathcal{Y})$. There exists a unique measurable map f^* of $(X,\mathcal{X})$ into $(Y,\mathcal{Y})$ such that $f^* \restriction X = f$.*

P r o o f . In fact, (i) and (ii) are dual to (i) and (iii) of Lemma 1.6, respectively. ☐

Theorem 2.3. MSID *is an epireflective subcategory of* MID.

P r o o f . Using Lemma 2.2, it is easy to verify that passing from an ID-measable space $(X,\mathcal{X})$ to the ID-measurable space $(X^*,\mathcal{X}^*)$ yields an epireflector of MID onto MSID. ☐

Definition 2.4. We say that $(X^*,\mathcal{X}^*)$ is the *sobrification* of $(X,\mathcal{X})$.

Corollary 2.5. SID *is a monocoreflective subcategory of* ID.

Definition 2.6. Let X be a non-empty set and let $\mathcal{X} \subseteq I^X$. Define

(i) $c(\mathcal{X}) = \{g \in I^X : g = \lim g_n,\ g_n \in \mathcal{X}\}$;

(ii) $o(\mathcal{X}) = \{\mathcal{Y} \subseteq I^X : \mathcal{X} \subseteq \mathcal{Y},\ \mathcal{Y} \in \mathrm{ID}\}$;

(iii) $d(\mathcal{X}) = \bigcap\limits_{\mathcal{Y} \in o(\mathcal{X})} \mathcal{Y}$;

(iv) for each ordinal number ξ define σ^ξ inductively:

(1) $\sigma^0(\mathcal{X}) = \mathcal{X}$,

(2) $\sigma^\xi = d\big(c\big(\sigma^{\xi-1}(\mathcal{X})\big)\big)$ if ξ is isolated,

(3) $\sigma^\xi = \bigcup\limits_{\eta<\xi} \sigma^\eta(\mathcal{X})$ if ξ is a limit ordinal, and denote

(4) $\sigma(\mathcal{X}) = \sigma^{\omega_1}(\mathcal{X})$.

EXAMPLE 2.7. Let $\mathbb{A}$ be a field of subsets of X. Let $\mathcal{X}$ be the set of all characteristic functions χ_A of sets belonging to $\mathbb{A}$. It is known that $c(\mathcal{X})$ is a field of subsets of X and $\sigma(\mathcal{X})$ is the set of all characteristic functions χ_A of sets belonging to the generated σ-field $\sigma(\mathbb{A})$.

EXAMPLE 2.8. Let $\mathcal{X} \subseteq I^X$ be a bold algebra. Then $c(\mathcal{X}) \subseteq I^X$ is a bold algebra and $\sigma(\mathcal{X})$ is the generated tribe (cf. Frič [15]).

LEMMA 2.9. *Let $(X,\mathcal{X})$ and $(Y,\mathcal{Y})$ be* ID*-measurable spaces and let h, h' be sequentially continuous* D*-homomorphisms of $\sigma(\mathcal{X})$ into $\sigma(\mathcal{Y})$ such that $h(g) = h'(g)$ for all $g \in \mathcal{X}$. Put $e(\mathcal{X}) = \{g \in \sigma(\mathcal{X}) \colon h(g) = h'(g)\}$. Then*

(i) *$e(\mathcal{X})$ is a sequentially closed subset of I^X;*

(ii) *$e(\mathcal{X})$ is a* D*-poset;*

(iii) *$h = h'$.*

Proof. (i) Let $\langle g_n \rangle$ be a sequence in $e(\mathcal{X})$ converging in I^X to g. Then $\langle h(g_n)\rangle$ converges in I^Y to $h(g)$ and $\langle h'(g_n)\rangle$ converges in I^Y to $h(g)$. Since $h(g_n) = h'(g_n)$ for all $n \in \mathbb{N}$ and the pointwise convergence in I^Y has unique limit, it follows that $h(g) = h'(g)$. Thus $e(\mathcal{X})$ is sequentially closed.

(ii) Clearly $0, 1 \in e(\mathcal{X})$. If $g_1, g_2 \in e(\mathcal{X})$ and $g_1 \le g_2$, then $g_2 \ominus g_1 \in \sigma(\mathcal{X})$, $h(g_1) \le h(g_2)$ and $h'(g_1) \le h'(g_2)$, and $h(g_2 \ominus g_1) = h(g_2) \ominus h(g_1) = h'(g_2) \ominus h'(g_1) = h'(g_2 \ominus g_1)$. Hence $g_2 \ominus g_1 \in e(\mathcal{X})$.

(iii) If follows from (i) and (ii) that $e(\mathcal{X})$ is a sequentially closed D-poset. Since $\mathcal{X} \subseteq e(\mathcal{X})$ and $\sigma(\mathcal{X})$ is the smallest sequentially closed containing $\mathcal{X}$, necessarily $e(\mathcal{X}) = \sigma(\mathcal{X})$. Thus $h = h'$. □

LEMMA 2.10.

(i) *$d(\mathcal{X})$ is a* D*-poset; it is the smallest* D*-poset in I^X which contains $\mathcal{X}$.*

(ii) *$\sigma^{\xi}(\mathcal{X})$ is a* D*-poset for each ordinal number ξ.*

(iii) *$\sigma^{\omega_1}(\mathcal{X})$ is sequentially closed; it is the smallest sequentially closed* D*-poset in I^X which contains $\mathcal{X}$.*

(iv) *$\sigma^{\omega_1+1}(\mathcal{X}) = \sigma^{\omega_1}(\mathcal{X})$.*

Proof. (i) and (ii) are trivial. (iii) follows from the order properties of ω_1. (iv) follows from (iii). □

Let $\mathcal{X} \subseteq I^X$ and $\mathcal{Y} \subseteq I^Y$ be D(I)-posets and let $f \colon Y \to X$ be an $(\mathcal{X},\mathcal{Y})$-measurable map.

LEMMA 2.11. (i) *$\mathcal{Z} = \{g \in I^X \colon g \circ f \in d(c(\mathcal{Y}))\}$ is a* D*-poset.*

(ii) *$d(c(\mathcal{X})) \subseteq \mathcal{Z}$.*

(iii) *$d(c(\mathcal{X})) \circ f \subseteq d(c(\mathcal{Y}))$.*

P r o o f. (i) Clearly $0_X, 1_X \in \mathcal{Z}$. Let $g_1, g_2 \in \mathcal{Z}$, $g_1 \le g_2$. Then $g_1 \circ f$, $g_2 \circ f \in d\big(c(\mathcal{Y})\big)$ and $g_1 \circ f \le g_2 \circ f$. Since $d\big(c(\mathcal{Y})\big)$ is a D-poset, $g_2 \circ f \ominus g_1 \circ f = (g_2 \ominus g_1) \circ f$ belongs to $d\big(c(\mathcal{Y})\big)$. Thus $g_2 \ominus g_1 \in \mathcal{Z}$.

(ii) Since f is measurable, we have $\mathcal{X} \subseteq \mathcal{Z}$. Let $g \in c(\mathcal{X})$ and let $\langle g_n \rangle$ be a sequence in $\mathcal{X}$ which converges in I^X to g. Then for each $y \in Y$ the sequence $\langle g_n\big(f(y)\big)\rangle$ converges to $g\big(f(y)\big)$. From $g_n \circ f \in \mathcal{Y}$, $n \in \mathbb{N}$, it follows that $g \circ f \in c(\mathcal{Y}) \subseteq d\big(c(\mathcal{Y})\big)$. Thus $g \in \mathcal{Z}$ and $c(\mathcal{X}) \subseteq \mathcal{Z}$. According to (i), $\mathcal{Z}$ is a D-poset and hence $d\big(c(\mathcal{Y})\big) \subseteq \mathcal{Z}$.

(iii) The assertion follows from (ii). □

Let η be an ordinal number, $\eta \le \omega_1$. For $g \in \sigma^\eta(\mathcal{X})$ put $h_\eta(g) = g \circ f$. This defines a map $h_\eta \colon \sigma^\eta(\mathcal{X}) \to I^Y$.

LEMMA 2.12. (i) *If $g \in \sigma^\eta(\mathcal{X})$, then $h_\eta(g) \in \sigma^\eta(\mathcal{Y})$.*

(ii) *$f \colon Y \to X$ is $\big(\sigma^\eta(\mathcal{X}),\ \sigma^\eta(\mathcal{Y})\big)$-measurable.*

P r o o f. Since $f \colon Y \to X$ is $(\mathcal{X}, \mathcal{Y})$-measurable, for $\eta = 0$ the assertions are trivial. Now, assume that the assertions are satisfied for all ordinal numbers less than η.

Let η be isolated. Then the assertion holds true for $\eta - 1$. This means that $f \colon Y \to X$ is $\big(\sigma^{\eta-1}(\mathcal{X}),\ \sigma^{\eta-1}(\mathcal{Y})\big)$-measurable. Now (i) follows from (iii) in Lemma 2.11 and (i) implies (ii).

Let η be a limit ordinal. Since $\sigma^\xi(\mathcal{X}) \circ f \subseteq \sigma^\xi(\mathcal{Y})$ for all $\xi < \eta$, necessarily $\sigma^\eta(\mathcal{X}) \circ f \subseteq \sigma^\eta(\mathcal{Y})$. Then f is $\big(\sigma^\eta(\mathcal{X}),\ \sigma^\eta(\mathcal{Y})\big)$-measurable. □

COROLLARY 2.13. *Let $\mathcal{X} \subseteq I^X$ and $\mathcal{Y} \subseteq I^Y$ be* ID*-objects and let $f \colon Y \to X$ be an $(\mathcal{X}, \mathcal{Y})$-measurable map. Then f is $\big(\sigma(\mathcal{X}),\ \sigma(\mathcal{Y})\big)$-measurable.*

DEFINITION 2.14. Let $\mathcal{X} \subseteq I^X$ be an ID-object. If $\mathcal{X} = c(\mathcal{X})$, then $\mathcal{X}$ is said to be *closed*. The measurable space $(X, \mathcal{X})$ is said to be *closed* if $\mathcal{X}$ closed.

Denote CID, resp. CSID, the (full) subcategory of ID the objects of which are closed, resp. sober, ID-objects. Denote CMID the (full) subcategory of MID the objects of which are closed measurable spaces.

The construction of $\sigma(\mathcal{X})$ and its properties lead to the following functor $\sigma_M \colon \mathrm{MID} \to \mathrm{CMID}$. If $(X, \mathcal{X})$ is a measurable space, then $\sigma_M(X, \mathcal{X}) = \big(X, \sigma(\mathcal{X})\big)$. Further, according to Corollary 2.13, each measurable map f of $(Y, \mathcal{Y})$ into $(X, \mathcal{X})$ is also measurable as the map of $\big(Y, \sigma(\mathcal{Y})\big)$ into $\big(X, \sigma(\mathcal{X})\big)$; denote it $\sigma_M(f)$. Clearly, the identity maps and compositions are preserved.

THEOREM 2.15. *σ_M is a monocoreflector.*

P r o o f. Let $(X, \mathcal{X})$ be a measurable space. Clearly, the identity map i_X as a measurable map of $\big(X, \sigma(\mathcal{X})\big)$ into $(X, \mathcal{X})$ is a monomorphism. Let f be a measurable map of a closed measurable space $(Y,\mathcal{Y})$ into $(X,\mathcal{X})$. Then f, as a map of $(Y,\mathcal{Y})$ into $\big(X,\sigma(\mathcal{X})\big)$ is measurable and, since we assume that all D(I)-posets are reduced, it is the unique measurable map of $(Y,\mathcal{Y})$ into $\big(X,\sigma(\mathcal{X})\big)$ such that its composition with i_X is equal to f. □

COROLLARY 2.16. CMID *is a monocoreflective subcategory of* MID.

Clearly, if $\mathcal{X}\subseteq I^X$ is a sober ID-object, then $\sigma(\mathcal{X})$ is sober, too.

COROLLARY 2.17. CSID *is an epireflective subcategory of* SID.

Proof. Let $\mathcal{X}\subseteq I^X$ and $\mathcal{Y}\subseteq I^Y$ be sober ID-objects and let h be a sequentially continuous D-homomorphisms of $\mathcal{X}$ into $\mathcal{Y}$. Since sober ID-objects and sober measurable spaces are dually isomorphic, there exists a unique measurable map f of $(Y,\mathcal{Y})$ into $(X,\mathcal{X})$ such that $f^{\triangleleft}=h$. According to Corollary 2.13, f is $\big(\sigma(\mathcal{X}),\sigma(\mathcal{Y})\big)$-measurable and hence induces a sequentially continuous D-homomorphism of $\sigma(\mathcal{X})$ into $\sigma(\mathcal{Y})$ which extends h. It follows from Lemma 2.9, that the extension is uniquely defined. The same Lemma 2.9 guarantees that the embedding of $\mathcal{X}$ into $\sigma(\mathcal{X})$ is an epimorphism. Thus passing from $\mathcal{X}$ to $\sigma(\mathcal{X})$ yields an epireflector of sober ID-objects to closed sober ID-objects. □

3. Applications

In this section we indicate possible applications of ID to probability.

EXAMPLE 3.1. Let $\mathbb{A}$ be a field of subsets of a set Ω and let $P(\mathbb{A})$ be the set of all probability measures on $\mathbb{A}$. Consider the evaluation map ev of $\mathbb{A}$ into $I^{P(\mathbb{A})}$ defined by $\mathrm{ev}(A)=\big\{p(A)\colon p\in P(\mathbb{A})\big\}$, $A\in\mathbb{A}$. As a rule, we identify $\omega\in\Omega$ and the corresponding point measure $\delta_\omega\colon\mathbb{A}\to I$, defined by $\delta_\omega(A)=1$ if $\omega\in A$ and $\delta_\omega(A)=0$ otherwise. It is easy to see that $\mathrm{ev}(\mathbb{A})=\big\{\mathrm{ev}(A)\colon A\in\mathbb{A}\big\}\subset I^{P(\mathbb{A})}$ is a D(I)-poset. Indeed, $\mathrm{ev}(\mathbb{A})$ carries the natural D-poset structure and each probability measures $p\in P(\mathbb{A})$ is sequentially continuous on $\mathbb{A}$ in the usual sequential convergence on $\mathbb{A}$ (cf. Novák [24], Frič [15]). If we identify $A\in\mathbb{A}$ and the corresponding characteristic function $\chi_A\colon X\to\{0,1\}\subset I$, then $\mathbb{A}$, $\mathrm{ev}(\mathbb{A})\restriction\Omega$, and $\mathrm{ev}(\mathbb{A})$ are isomorphic D(I)-posets. Clearly, $\mathbb{A}$ is reduced iff $\mathrm{ev}(\mathbb{A})$ is reduced. In what follows, we identify $\mathbb{A}$ and $\mathrm{ev}(\mathbb{A})\restriction\Omega$. Observe that, except for trivial cases, $\mathrm{ev}(\mathbb{A})$ fails to be a bold algebra (i.e., a semisimple MV-algebra). Let $\mathbb{A}$ be reduced and let X be a subset of $P(\mathbb{A})$ such that $\Omega\subseteq X$. Then $\mathcal{X}=\mathrm{ev}(\mathbb{A})\restriction X\subseteq I^X$ is a reduced D(I)-poset isomorphic to $\mathrm{ev}(\mathbb{A})$ and $(X,\mathcal{X})$ is a dense subspace of the ID-measurable space $\big(P(\mathbb{A}),\mathrm{ev}(\mathbb{A})\big)$.

Definition 3.2. Let $\mathbb{A}$ be a field of subsets of Ω and let $\Omega \subseteq X \subseteq P(\mathbb{A})$. Then the points of X are said to be *extended elementary events*, $\mathcal{X} = \mathrm{ev}(\mathbb{A}) \restriction X \subseteq I^X$ is said to be an *extended field of events*, and $(X, \mathcal{X})$ is said to be an *extended measurable space*.

Lemma 3.3. *Let $\mathbb{A}$ be a reduced field of subsets of Ω and let $\mathcal{X} = \mathrm{ev}(\mathbb{A}) \restriction X \subseteq I^X$, $\Omega \subseteq X \subseteq P(\mathbb{A})$, be an extended field of events. Let $h\colon \mathcal{X} \to I$ be a sequentially continuous* D*-homomorphism. Then there exists a unique probability measure $p \in P(\mathbb{A})$ such that $p(A) = h\big(\mathrm{ev}(A) \restriction \Omega\big)$ for all $A \in \mathcal{A}$.*

Proof. Remember, $(\Omega, \mathcal{A})$ is a dense subspace of $(X, \mathcal{X})$ and hence for $a, b \in \mathcal{X}$, $a \restriction \Omega \neq b \restriction \Omega$ implies $a \neq b$. Consequently $h(a) = q(a \restriction \Omega)$, $a \in \mathcal{X}$, defines a unique sequentially continuous D-homomorphism q of $\mathbb{A}$ into I. Clearly, $q(\emptyset) = 0$, $q(\Omega) = 1$, $q(A) \le q(B)$ whenever $A, B \in \mathbb{A}$ and $A \subseteq B$. If $A \cap B = \emptyset$, then $\mathrm{ev}(A \cup B) = \big\{p(A) + p(B)\colon p \in P(\mathbb{A})\big\}$ and hence $\big(\mathrm{ev}(A \cup B) \restriction X\big) \ominus \big(\mathrm{ev}(B) \restriction X\big) = \mathrm{ev}(A) \restriction X$. Thus $h\big(\mathrm{ev}(A) \restriction X\big) + h\big(\mathrm{ev}(B) \restriction X\big) = q(A) + q(B)$. This means that q is a sequentially continuous finitely additive measure on $\mathbb{A}$ and hence a probability measure on $\mathbb{A}$ (cf. Frič [15]). $\square$

Corollary 3.4. *Let $\mathbb{A}$ be a reduced field of subsets of Ω. Then $\mathrm{ev}(\mathbb{A}) \subseteq I^{P(\mathbb{A})}$ is a reduced sober* ID*-object.*

Definition 3.5. Let $\mathcal{X} = \mathrm{ev}(\mathbb{A}) \restriction X \subseteq I^X$, $\Omega \subseteq X \subseteq P(\mathbb{A})$, be an extended field of events and let $h\colon \mathcal{X} \to I$ be a sequentially continuous D-homomorphism. Then h is said to be an *extended probability* on $\mathcal{X}$.

Example 3.6. Let $\mathbb{A}$ be a reduced field of subsets of Ω, let $\mathbb{B}$ be a reduced field of subsets of Θ, let $P(\mathbb{A})$ and $P(\mathbb{B})$ be the probability measures on $\mathbb{A}$ and $\mathbb{B}$, respectively. Let $f\colon \Omega \to \Theta$ be a $(\mathbb{B}, \mathbb{A})$-measurable map. As in Example 3.1, consider the ID-measurable spaces $\big(P(\mathbb{A}), \mathrm{ev}(\mathbb{A})\big)$ and $\big(P(\mathbb{B}), \mathrm{ev}(\mathbb{B})\big)$. Let $p \in P(\mathbb{A})$. Consider the preimage map $f^{\leftarrow}$ sending $B \in \mathbb{B}$ into $f^{\leftarrow}(B) = \big\{\omega \in \Omega\colon f(\omega) \in B\big\} \in \mathbb{A}$. Then $p \circ f^{\leftarrow}$ is a probability on $\mathbb{B}$. Let f_P be the resulting map of $P(\mathbb{A})$ into $P(\mathbb{B})$. Clearly, $f_P \restriction \Omega = f$ and f_P is $\big(\mathrm{ev}(\mathbb{B}), \mathrm{ev}(\mathbb{A})\big)$-measurable. Let $\sigma(\mathbb{A})$ and $\sigma(\mathbb{B})$ be the generated σ-fields. Since each probability measure on a field of sets can be uniquely extended to a probability measure on the generated σ-field, we can identify probability measures on $\mathbb{A}$ and $\sigma(\mathbb{A})$, resp. on $\mathbb{B}$ and $\sigma(\mathbb{B})$. Then f_P is $\big(\mathrm{ev}\big(\sigma(\mathbb{B})\big), \mathrm{ev}\big(\sigma(\mathbb{A})\big)\big)$-measurable. Let $\sigma\big(\mathrm{ev}(\mathbb{A})\big)$ and $\sigma\big(\mathrm{ev}(\mathbb{B})\big)$ be the subsets of $I^{P(\mathbb{A})}$ and $I^{P(\mathbb{B})}$ describes in Definition 2.6. It follows from Corollary 2.13 that f_P is $\big(\sigma\big(\mathrm{ev}(\mathbb{B})\big), \sigma\big(\mathrm{ev}(\mathbb{A})\big)\big)$-measurable. Observe that the evaluation map ev sending $\mathbb{A}$ onto $\mathrm{ev}(\mathbb{A}) \subseteq I^{P(\mathbb{A})}$ has been used by J. Novák to construct the P-sequential envelope and to prove that the P-sequential envelope of $\mathbb{A}$ is $\sigma(\mathbb{A})$ (cf. Novák [25]).

LEMMA 3.7. *Let $\mathbb{A}$ be a field of subsets of Ω, let $\sigma(\mathbb{A})$ be the generated σ-field, let $P(\mathbb{A}) = P(\sigma(\mathbb{A}))$ be the probability measures on $\mathbb{A}$ and $\sigma(\mathbb{A})$, let $(P(\mathbb{A}), \mathrm{ev}(\mathbb{A}))$ and $(P(\mathbb{A}), \mathrm{ev}(\sigma(\mathbb{A})))$ be the corresponding* ID*-measurable spaces, and let $(P(\mathbb{A}), \sigma(\mathrm{ev}(\mathbb{A})))$ be the σ_M-coreflection of $(P(\mathbb{A}), \mathrm{ev}(\mathbb{A}))$. Then $\sigma(\mathrm{ev}(\mathbb{A})) = \mathrm{ev}(\sigma(\mathbb{A}))$.*

Proof. Clearly, $\mathrm{ev}(\sigma(\mathbb{A}))$ contains $\mathrm{ev}(\mathbb{A})$. According to (iii) in Lemma 2.10, $\sigma(\mathrm{ev}(\mathbb{A}))$ is the smallest sequentially closed D-poset in $I^{P(\mathbb{A})}$ containing $\mathrm{ev}(\mathbb{A})$. As proved in Novák [25], $\mathrm{ev}(\sigma(\mathbb{A}))$ is the smallest sequentially closed subset of $I^{P(\mathbb{A})}$ containing $\mathrm{ev}(\mathbb{A})$, hence $\mathrm{ev}(\sigma(\mathbb{A}))$ is a subset of $\sigma(\mathrm{ev}(\mathbb{A}))$. Since $\mathrm{ev}(\sigma(\mathbb{A}))$ is a D-poset, $\sigma(\mathrm{ev}(\mathbb{A}))$ is a subset of $\mathrm{ev}(\sigma(\mathbb{A}))$. Thus $\sigma(\mathrm{ev}(\mathbb{A})) = \mathrm{ev}(\sigma(\mathbb{A}))$. □

Example 3.6 leads to a functor sending each reduced classical measurable space $(\Omega, \mathbb{A})$ into the sober extended measurable space $(P(\mathbb{A}), \mathrm{ev}(\mathbb{A}))$. Put $\varepsilon(\Omega, \mathbb{A}) = (P(\mathbb{A}), \mathrm{ev}(\mathbb{A}))$ and denote ε the resulting functor. Denote σ the functor sending each classical measurable space $(\Omega, \mathbb{A})$ into $(\Omega, \sigma(\mathbb{A}))$. Now Lemma 3.7 implies the following corollary.

COROLLARY 3.7. $\varepsilon \circ \sigma = \sigma_M \circ \varepsilon$.

Consider the situation described in Example 3.6. Observe that f_P maps points of Ω into points of Θ. There are successful attempts (cf. Bugajski [3], [4], Gudder [17]) to generalize the classical Kolmogorovian notion of a random variable to suitable maps of $P(\mathbb{A})$ into $P(\mathbb{B})$ in such a way that $f(\omega)$, $\omega \in \Omega$, can be a nondegenerated probability measure on $\mathbb{B}$. Such maps are called operational random variables and have definite quantum nature. In Frič [16] it is shown that operational random variables and observables (as their dual notions) can be described, and further generalized, within ID and MID in a natural way.

EXAMPLE 3.8. Let $\mathcal{X}_0 \subseteq I^X$ be a reduced D(I)-poset and let $\mathcal{X} = \sigma(\mathcal{X}_0) \subseteq I^X$ be the epireflection of $\mathcal{X}_0$ into CID. Then each sequentially continuous D-homomorphism p_0 of $\mathcal{X}_0$ into I can be uniquely extended to a sequentially continuous D-homomorphism p over $\mathcal{X}$. Let $\mathcal{M} \subseteq I^R$ be the ID-objects consisting of all Borel measurable real functions into I, consider R as the subset of the set R^* of all sequentially continuous D-homomorphisms of $\mathcal{M}$ into I, and let $\mathcal{R} \subset I^{R^*}$ the evaluation of $\mathcal{M}$. In view of the results of Section 1 and Section 2, it is naturel to consider $\mathcal{X}$ as generalized events, each $(\mathcal{R}, \mathcal{X})$-measurable map $f\colon X \to R^*$ as a generalized random variable, and each sequentially continuous D-homomorphism $h\colon \mathcal{R} \to \mathcal{X}$ (can be represented by a unique $f^{\triangleleft}$) as a generalized observable. Finally each sequentially continuous D-homomorphism p on $\mathcal{X}$ into I can be considered as a generalized probability measure on $\mathcal{X}$.

4. Infinitesimals

In this section we modify some of the previous results replacing the closed unit interval I by an ultrapower I_u and replacing the sequential convergence by approximation. We generalize the corresponding constructions in Frič [13] from bold algebras to $\mathrm{D}(I_u)$-posets.

For the reader's convenience, recall some notions related to nonstandard numbers. Let U be a nonempty set and let u be an ultrafilter on U. Consider the set R^U of all maps of U into R carrying the natural equivalence relation: $r \sim_u s$ whenever there exists a subset X of U such that $X \in u$ and $X = \{x \in U : r(x) = s(x)\}$. It is known that the usual order and the algebraic operations on real numbers can be extended over the quotient set $R^U/_{\sim_u} = R_u$. This yields a nonstandard model of real numbers. Equivalence classes containing constant maps lead to the embedding of R into R_u. To each subset A of R there corresponds a unique subset A_u of R_u: $a \in A_u$ iff there exists $b \in A^U$ such that a is the equivalence class containing b. If a positive nonstandard number a is less than each positive real number, then a is said to be a positive infinitesimal. To each bounded number $r \in R_u$ there corresponds a unique real number $\mathrm{st}(r)$, called the standard part of r, such that $r - \mathrm{st}(r)$ is an infinitesimal.

It is known (cf. Di Nola [7]) that for each MV-algebra M there exists an ultrafilter u and a set X such that M is isomorphic to a subalgebra $\mathcal{X}$ of the power MV-algebra I_u^X, where I_u is the subset of R_u corresponding to I. If u is a fixed ultrafilter then $I = I_u$ and $\mathcal{X} \subseteq I^X$; in this case M is said to be semisimple (Archimedean). Since I_u^X carries a natural D-poset structure, it is natural to define $\mathrm{D}(I_u)$-posets and to generalize some of the results of Section 2 and Section 3 to $\mathrm{D}(I_u)$-posets. Observe that there is no suitable sequential convergence in I_u and hence in I_u^X.

Definition 4.1. For $\{a(x) : x \in X\} \in I_u^X$ define $\mathrm{st}\big(\{a(x) : x \in X\}\big) = \big\{\mathrm{st}\big(a(x)\big) : x \in X\big\}$. We say that a sequence $\langle a_n \rangle$ in I_u^X *approximates* $a \in I_u^X$ if the sequence $\langle \mathrm{st}(a_n) \rangle$ converges in I^X pointwise to $\mathrm{st}(a)$. A map h of $\mathcal{X} \subseteq I_u^X$ into $\mathcal{Y} \subseteq I_u^Y$ is said to be *approximately continuous* if whenever a sequence $\langle a_n \rangle$ in $\mathcal{X}$ approximates $a \in \mathcal{X}$, then the sequence $\langle h(a_n) \rangle$ in $\mathcal{Y}$ approximates $h(a)$.

Clearly, each evaluation $\mathrm{ev}_x : \mathcal{X} \to I_u$, $x \in X$, is approximately continuous.

Let u be a free ultrafilter, let $\mathbb{R}_u$ be the corresponding nonstandard real line. Consider $I_u = \{x \in \mathbb{R}_u : 0 \le x \le 1\}$ as a D-poset (the order and difference are inherited from $\mathbb{R}_u$) and, for each set X, consider the product D-poset I_u^X (the order and operations are defined pointwise).

Definition 4.2. A subset $\mathcal{X} \subseteq I_u^X$, containing the constant functions $0_{\mathcal{X}}$, $1_{\mathcal{X}}$ and closed with respect to the difference inherited from I_u^X, is said to be a $\mathrm{D}(I_u)$-*poset*. If for each approximately continuous D-homomorphism h of $\mathcal{X}$ into I_u there exists $x \in X$ such that $h = \mathrm{ev}_x$, then $\mathcal{X}$ is said to be *sober*.

Let $\mathrm{I}_u\mathrm{D}$ be be the category the objects of which are reduced $\mathrm{D}(I_u)$-posets and morphisms of which are approximately continuous D-homomorphisms. Let $\mathrm{SI}_u\mathrm{D}$ be the (full) subcategory of $\mathrm{I}_u\mathrm{D}$ consisting of reduced sober objects.

Definition 4.3. Let $\mathcal{X} \subseteq I_u^X$ be a $\mathrm{D}(I_u)$-poset. Then $(X, \mathcal{X})$ is said to be a $\mathrm{D}(I_u)$-*measurable space*. If $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ are $\mathrm{D}(I_u)$-measurable spaces and f is a map of Y into X such that $a \circ f \in \mathcal{Y}$ for all $a \in \mathcal{X}$, i.e., $\mathcal{X} \circ f \subseteq \mathcal{Y}$, then f is said to be $(\mathcal{X}, \mathcal{Y})$-*measurable*, or simply *measurable*. If $\mathcal{X}$ is sober or reduced, then $(X, \mathcal{X})$ is said to be *sober* or *reduced*.

In $\mathrm{I}_u\mathrm{D}$, let f be an $(\mathcal{X}, \mathcal{Y})$-measurable map and let $f^{\triangleleft}$ be the usual map of $\mathcal{X}$ into $\mathcal{Y}$. We leave out the obvious proof of next two lemmas.

Lemma 4.4. *$f^{\triangleleft}$ is an approximately continuous* D*-homomorphism.*

Lemma 4.5. *Let h be an approximately continuous* D*-homomorphism of $\mathcal{X}$ into $\mathcal{Y}$. If $\mathcal{X}$ is sober, then there exists a unique $(\mathcal{X}, \mathcal{Y})$-measurable map g such that $h = g^{\triangleleft}$.*

Observe that the assumption of soberness is essential.

Let $\mathrm{MI}_u\mathrm{D}$ be the category the objects of which are reduced $\mathrm{D}(I_u)$-measurable spaces and the morphisms of which are approximately continuous $\mathrm{D}(I_u)$-measurable maps. Let $\mathrm{SMI}_u\mathrm{D}$ be the (full) subcategory of $\mathrm{MI}_u\mathrm{D}$ consisting of sober objects.

Theorem 4.6. *The categories $\mathrm{SI}_u\mathrm{D}$ and $\mathrm{MSI}_u\mathrm{D}$ are dually isomorphic.*

P r o o f. It follows from Lemma 4.4 and Lemma 4.5 that passing from a reduced sober $\mathrm{D\,I}_u$-poset $\mathcal{X} \subseteq I_u^X$ to the $\mathrm{D}(I_u)$-measurable space $(X, \mathcal{X})$ and back, resp. passing from an approximately continuous D-homomorphism h to the unique measurable map f such that $f^{\triangleleft} = h$, yields the desired dual isomorphism. □

Let $\mathcal{X} \subseteq I_u^X$ be a reduced $\mathrm{D}(I_u)$-poset. If $\mathcal{X}$ fails to be sober, consider X as the subset of the set X^* of all approximately continuous D-homomorphisms of $\mathcal{X}$ into I_u and for $a \in \mathcal{X}$ define $a^* \subseteq I_u^{X^*}$ as follows: $a^*(x) = x(a)$, $x \in X^*$. Let $\mathcal{X}^* = \{a^* \in I_u^{X^*} : a \in \mathcal{X}\}$. Then, virtually in the same way as in Section 1 and Section 2 we can prove that the relationships between $\mathcal{X}$ and $\mathcal{X}^*$ yield the next three propositions. We leave out details.

Theorem 4.7. (i) *The categories $\mathrm{I}_u\mathrm{D}$ and $\mathrm{SI}_u\mathrm{D}$ are naturally equivalent.*

(ii) *The categories $\mathrm{I}_u\mathrm{D}$ and $\mathrm{MSI}_u\mathrm{D}$ are dually equivalent.*

THEOREM 4.8. $\mathrm{SI}_u\mathrm{D}$ *is an monocoreflective subcategory of* $\mathrm{I}_u\mathrm{D}$.

THEOREM 4.9. $\mathrm{MSI}_u\mathrm{D}$ *is an epireflective subcategory of* $\mathrm{MI}_u\mathrm{D}$.

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