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# THE SPACE OF THE STRONG ŚWIĄTKOWSKI FUNCTIONS

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ABSTRACT. The definition of a strong Świątkowski property for a function $f\colon [a,b] \to \mathbb{R}$ at a fixed point of $[a,b]$ has been formulated in the paper [J. Kucner, R. J. Pawlak: *On local characterization of the strong Świątkowski property for a function* $f\colon [a,b] \to \mathbb{R}$, Real Anal. Exchange **28** (2003), 563–572]. Theorem 11 presented in that paper seems to suggest the question connected with comparisons of the class of strong Świątkowski functions with the Baire class one.

In this paper we give some answers to this question. In particular, we prove that in the space of bounded strong Świątkowski functions (with the metric of uniform convergence) a set $\mathcal{L}$ of all functions measurable (in the Lebesgue sense) is a superporous set at each point of this space.

In 1977 T. Mańk and T. Świątkowski have introduced a new class of functions ([4]) (a function belonging to this class is called the Świątkowski function [6]). A. Maliszewski has modified this definition ([3]) and introduced the notion of the strong Świątkowski property. He noticed that functions having this property are Darboux and quasi-continuous functions. In [2] the idea of local characterization of the strong Świątkowski property (at a fixed point) was introduced.

In this paper we apply the classical symbols and notions.

The symbol $(a,b)$ denotes an open interval in the case if $a<b$ or $b<a$ (we apply the similar assumption for other intervals). However, in the case when $(a,b)$ or $[a,b]$ is the domain of some functions, we assume that $a<b$.

If $(X,d)$ is a metric space then $B(x,r)$ denotes the open ball centered at $x$ with radius $r>0$. Let $M \subset X$, $x \in X$, $R>0$. Then $\gamma(x,R,M)$ denotes the supremum of the set of all $r>0$ for which there exists $z \in X$ such that $B(z,r) \subset B(x,R) \setminus M$. The set $M$ is called porous at $x$ if $\limsup\limits_{R\to 0+} \frac{\gamma(x,R,M)}{R} > 0$.

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We say that $E \subset X$ is superporous at $x \in X$ provided that $E \cup F$ is porous at $x$ for any set $F$ porous at $x$.

Let $f\colon X \to \mathbb{R}$ ($X$ is a connected subset of $\mathbb{R}$). By

1° $\mathrm{L}^-(f,x) = \{y \in \overline{\mathbb{R}}\colon \exists_{x_n \to x,\, x_n < x} \lim_{n\to\infty} f(x_n) = y\}$,

2° $\mathrm{L}^+(f,x) = \{y \in \overline{\mathbb{R}}\colon \exists_{x_n \to x,\, x_n > x} \lim_{n\to\infty} f(x_n) = y\}$,

3° $\mathrm{L}(f,x) = \{y \in \overline{\mathbb{R}}\colon \exists_{x_n \to x,\, x_n \neq x} \lim_{n\to\infty} f(x_n) = y\}$,

we denote: the cluster set from the left, the cluster set from the right and the cluster set of the function $f$ at the point $x \in X$.

Let $f$ be a function. We denote:

- by $\mathcal{C}(f)$ ($\mathcal{C}^+(f)$, $\mathcal{C}^-(f)$) the set of all bilaterally (right-hand, left-hand) continuity points of the function $f$;
- by $\mathcal{D}(f)$ the set of all discontinuity points of the function $f$.

The symbol $\lambda$ stands for the Lebesgue measure on the real line.

A function $f\colon [a,b] \to \mathbb{R}$ is the strong Świątkowski function ([3]) if for any two elements $x, y \in [a,b]$ such that $f(x) \neq f(y)$ and for arbitrary $\eta \in \big(f(x), f(y)\big)$ there exists a point $z \in (x,y) \cap \mathcal{C}(f)$ such that $f(z) = \eta$.

By $\mathcal{S}$ we denote the set of all functions $f\colon [a,b] \to \mathbb{R}$ having the strong Świątkowski property.

By $\mathrm{b}\mathcal{S}$ we denote the set of all bounded functions $f\colon [a,b] \to \mathbb{R}$ having the strong Świątkowski property.

Let $\{A_s\}_{s\in S}$ be a covering of topological space $X$ and let $f_s$ be a family of functions $f_s\colon A_s \to Y$. We say that functions $f_s$ ($s \in S$) are compatible if for every pair $s_1$, $s_2$ of elements of $S$ we have $f_{s_1}|A_{s_1} \cap A_{s_2} = f_{s_2}|A_{s_1} \cap A_{s_2}$; letting $f(x) = f_s(x)$ for $x \in A_s$ we define then a function $f$ of $X$ to $Y$, which is called the combination of functions $\{f_s\}_{s\in S}$ and is denoted by the symbol $\nabla_{s\in S} f_s$ ([1]).

**Definition 1.** ([2]) Let $f\colon [a,b] \to \mathbb{R}$. We say that a *point* $x_0 \in (a,b)$ *cuts a function* $f$, if there exists a number $\delta > 0$ such that $(x_0 - \delta,\, x_0 + \delta) \subset (a,b)$ and either

$$f\big((x_0-\delta, x_0) \cap \mathcal{C}(f)\big) \subset \big(-\infty, f(x_0)\big) \text{ and } f\big((x_0, x_0+\delta) \cap \mathcal{C}(f)\big) \subset \big(f(x_0), +\infty\big)$$

or

$$f\big((x_0-\delta, x_0) \cap \mathcal{C}(f)\big) \subset \big(f(x_0), +\infty\big) \text{ and } f\big((x_0, x_0+\delta) \cap \mathcal{C}(f)\big) \subset \big(-\infty, f(x_0)\big).$$

**Definition 2.** ([2]) Let $f\colon [a,b] \to \mathbb{R}$. We say that a function $f$ has a *right-hand strong Świątkowski property at a point* $x_0 \in [a,b)$ (briefly $x_0 \in \mathcal{S}^+(f)$) if either $x_0 \in \mathcal{C}^+(f)$ or

1° for any $\alpha \in \mathrm{L}^{+}(f, x_0) \setminus \{f(x_0)\}$ and $\beta \in (\alpha, f(x_0))$ and arbitrary positive number $\delta$ such that $(x_0, x_0 + \delta) \subset (a, b)$

$$f^{-1}(\beta) \cap (x_0, x_0 + \delta) \cap \mathcal{C}(f) \neq \emptyset, \text{ and}$$

2° for any $\alpha \in \mathbb{R}$: if there exist sequences $\{x_n\}, \{y_n\} \subset (a, b)$ such that $x_n \searrow x_0 \swarrow y_n$ and $f(x_n) \searrow \alpha \nwarrow f(y_n)$, then there exists a sequence $\{z_n\} \subset (a, b)$ such that $\{z_n\} \subset \mathcal{C}(f)$, $z_n \searrow x_0$ and $f(z_n) = \alpha$ for any $n \in \mathbb{N}$.

**Definition 3.** ([2]) Let $f\colon [a, b] \to \mathbb{R}$. We say that a function $f$ has a *left-hand strong Świątkowski property at a point* $x_0 \in (a, b]$ (briefly $x_0 \in \mathcal{S}^{-}(f)$) if either $x_0 \in \mathcal{C}^{-}(f)$ or

1° for any $\alpha \in \mathrm{L}^{-}(f, x_0) \setminus \{f(x_0)\}$ and $\beta \in (\alpha, f(x_0))$ and arbitrary positive number $\delta$ such that $(x_0 - \delta, x_0) \subset (a, b)$

$$f^{-1}(\beta) \cap (x_0 - \delta, x_0) \cap \mathcal{C}(f) \neq \emptyset, \text{ and}$$

2° for any $\alpha \in \mathbb{R}$: if there exist sequences $\{x_n\}, \{y_n\} \subset (a, b)$ such that $x_n \nearrow x_0 \nwarrow y_n$ and $f(x_n) \searrow \alpha \nwarrow f(y_n)$, then there exists a sequence $\{z_n\} \subset (a, b)$ such that $\{z_n\} \subset \mathcal{C}(f)$, $z_n \nearrow x_0$ and $f(z_n) = \alpha$ for any $n \in \mathbb{N}$.

**Definition 4.** ([2]) We say that a function $f\colon [a, b] \to \mathbb{R}$ has a *strong Świątkowski property at a point* $x_0 \in [a, b]$ (briefly $x_0 \in \mathcal{S}(f)$):

1° $x_0 \in \mathcal{S}^{+}(f)$, for $x_0 \in [a, b)$;

2° $x_0 \in \mathcal{S}^{-}(f)$, for $x_0 \in (a, b]$;

moreover, $f$ must be continuous at each point of $(a, b)$ that cuts $f$.

Before we stating the main result of this paper, we will recall some remarks.

**Remark 5.** ([2]) Let $f\colon [a, b] \to \mathbb{R}$. If $x_0 \in [a, b]$ is a point of continuity of $f$, then $f$ has a strong Świątkowski property at a point $x_0$.

**Theorem 6.** ([2], Theorem 11) *Let $f\colon [a, b] \to \mathbb{R}$. If $f$ has a strong Świątkowski property, then $\mathcal{D}(f)$ is a first category set.*

**Theorem 7.** ([2]) *A function $f\colon [a, b] \to \mathbb{R}$ is a strong Świątkowski function if and only if $[a, b] \subset \mathcal{S}(f)$.*

Let $\{a_i\}_{i \in K}$ ($K = \{1, 2, \ldots, n-1\}$) be an increasing and finite sequence of real numbers from an interval $(a, b)$ ($a < b$). Put $F_1 = [a, a_1]$, $F_i = [a_{i-1}, a_i]$ for $i \in K \setminus \{1\}$, $F_n = [a_{n-1}, b]$. Obviously, the family $\{F_i\}_{i \in K}$ is a closed covering of $[a, b]$.

**Lemma 8.** *Let* $\{F_i\}_{i\in K}$ *be a sequence of the intervals defined above and let* $f_i\colon F_i \to \mathbb{R}$ *(for* $i \in K$*) be strong Świątkowski functions that satisfy the following condition: if* $a_i$ *(*$i \in K$*) cuts* $f = \nabla_{i\in K} f_i$*, then* $a_i \in \mathcal{C}(f)$*.*

*Thus* $f$ *is a strong Świątkowski function.*

P r o o f. Let $x_0 \in [a,b]$. We shall show that $x_0 \in \mathcal{S}(f)$.

If $x_0 \in \mathcal{C}(f)$, then (Remark 5) $x_0 \in \mathcal{S}(f)$.

Thus we suppose that $x_0 \notin \mathcal{C}(f)$. We will show that

$$x_0 \in \mathcal{S}^+(f). \tag{1}$$

Of course, if $x_0 \in \mathcal{C}^+(f)$, then $x_0 \in \mathcal{S}^+(f)$ (see Definition 2), so, we suppose that $x_0 \notin \mathcal{C}^+(f)$. Then $x_0 \in [a_{i_0-1}, a_{i_0}) \subset F_{i_0}$ or $x_0 \in [a, a_1) \subset F_1$ or $x_0 \in [a_{n-1}, b) \subset F_n$.

Assume, for instance, that $x_0 \in [a_{i_0-1}, a_{i_0}) \subset F_{i_0}$, for $i_0 \in K \setminus \{1\}$ (other cases can be proved in a similar way). Now, to prove (1), we check the conditions of the Definition 2.

$1^\circ$ Let $\alpha \in \mathrm{L}^+(f, x_0) \setminus \{f(x_0)\}$. Then we can deduce that $\alpha \in \mathrm{L}^+(f_{i_0}, x_0) \setminus \{f_{i_0}(x_0)\}$. Suppose, for instance, that $\alpha < f(x_0)$. Let $\beta \in (\alpha, f(x_0)) = (\alpha, f_{i_0}(x_0))$ and let $\delta > 0$. Thus we have

$$f^{-1}(\beta) \cap (x_0, x_0 + \delta) \cap \mathcal{C}(f) \supset f_{i_0}^{-1}(\beta) \cap (x_0, x_0 + \delta) \cap \operatorname{int} F_{i_0} \cap \mathcal{C}(f_{i_0}).$$

Since $x_0$ is not the right end-point of the interval $F_{i_0}$, then $(x_0,\, x_0 + \delta) \cap \operatorname{int} F_{i_0} = (x_0,\, x_0 + \delta')$, for some $\delta' > 0$. So, ($f_{i_0} \in \mathcal{S}$),

$$f_{i_0}^{-1}(\beta) \cap (x_0, x_0 + \delta) \cap \operatorname{int} F_{i_0} \cap \mathcal{C}(f_{i_0}) = f_{i_0}^{-1}(\beta) \cap (x_0, x_0 + \delta') \cap \mathcal{C}(f_{i_0}) \neq \emptyset.$$

$2^\circ$ Let $\{x_n\}$, $\{y_n\}$ be sequences such that $x_n \searrow x_0 \swarrow y_n$ and $f(x_n) \searrow f(x_0) \nwarrow f(y_n)$.

Since $x_0$ is not the right end-point of the interval $F_{i_0}$, then $x_n \in F_{i_0}$ and $y_n \in F_{i_0}$ for sufficiently large $n \in \mathbb{N}$. We have $f(x_n) = f_{i_0}(x_n)$ and $f(y_n) = f_{i_0}(y_n)$. Hence $f_{i_0}(x_n) \searrow f(x_0) \swarrow f_{i_0}(y_n)$. Since $f_{i_0} \in \mathcal{S}$, then (Theorem 7, Definition 2) there exists a sequence $\{z_n\} \subset \mathcal{C}(f_{i_0})$ such that $z_n \searrow x_0$ and $f_{i_0}(z_n) = f_{i_0}(x_0)$. Thus we have $z_n \in \operatorname{int} F_{i_0} \cap \mathcal{C}(f_{i_0}) \subset \mathcal{C}(f)$, for each $n \in \mathbb{N}$. Moreover, $f(z_n) = f_{i_0}(z_n) = f_{i_0}(x_0) = f(x_0)$, for each $n \in \mathbb{N}$.

So, we can deduce that $x_0 \in \mathcal{S}^+(f)$, which finishes the proof of (1).

Obviously, if $x_0 \in \mathcal{C}^-(f)$, then $x_0 \in \mathcal{S}^-(f)$ (from Definition 3). Now, we assume that $x_0 \notin \mathcal{C}^-(f)$. Thus $x_0 \in (a_{i_0-1}, a_{i_0}] \subset F_{i_0}$, for $i_0 \in K \setminus \{1\}$ or $x_0 \in (a, a_1] \subset F_1$ or $x_0 \in (a_{n-1}, b] \subset F_n$. In a way similar to the above one, we show that $x_0 \in \mathcal{S}^-(f)$.

To finish the proof of the Lemma we need to show that for $x_0 \in (a,b)$, if $x_0$ cuts $f$ then it is the continuity point of $f$.

Without a loss of generality, we suppose that there exists $\delta^* > 0$ such that

$$f\big((x_0-\delta^*, x_0) \cap \mathcal{C}(f)\big) \subset \big(-\infty, f(x_0)\big)$$

and

$$f\big((x_0,\, x_0+\delta^*) \cap \mathcal{C}(f)\big) \subset \big(f(x_0), +\infty\big)\,.$$

We consider the following cases:

I. If there exists $i_0 \in K \setminus \{1\}$ such that $x_0 \in \operatorname{int} F_{i_0}$ (if $x_0 \in \operatorname{int} F_1$ or $x_0 \in \operatorname{int} F_n$ the proof is similar), then we may assume that $\delta^* > 0$ is a real number such that $(x_0-\delta^*, x_0+\delta^*) \subset \operatorname{int} F_{i_0}$. Therefore,

$$f_{i_0}\big((x_0-\delta^*, x_0) \cap \mathcal{C}(f_{i_0})\big) \subset \big(-\infty, f_{i_0}(x_0)\big)$$

and

$$f_{i_0}\big((x_0,\, x_0+\delta^*) \cap \mathcal{C}(f_{i_0})\big) \subset \big(f_{i_0}(x_0), +\infty\big)\,.$$

So, we can deduce that $x_0$ cuts the function $f_{i_0}$. Since $f_{i_0} \in \mathcal{S}$, then $x_0 \in \mathcal{C}(f_{i_0})$ and $x_0 \in \mathcal{C}(f)$.

II. Let $x_0 = a_{i_0}$ for some $i_0 \in K$. Since $x_0$ cuts the function $f$, then from the assumption of this Lemma we have $x_0 \in \mathcal{C}(f)$, which finishes the proof of the fact that $x_0 \in \mathcal{S}(f)$. $\square$

It is easy to see that there exists a Baire one function $f$ which does not have the strong Świątkowski property (consider, e.g., $\chi_{\{0\}}$—the characteristic function of a set $\{0\}$). It turns out that there exist functions $f \in \mathcal{S} \setminus \mathcal{B}_1$. This fact follows from the next (more stronger) theorem.

**Theorem 9.** *In the space* b$\mathcal{S}$ *(with the metric of uniform convergence) a set* $\mathcal{L}$ *of all measurable (in the Lebesgue sense) functions is a superporous set at each point of this space.*

P r o o f. Let $f \in \mathrm{b}\mathcal{S}$ and let $A \subset \mathrm{b}\mathcal{S}$ be a porous set at $f$. Moreover, let $R > 0$ and $r_1' = \frac{\gamma(f,R,A)}{2} > 0$. Then there exist $r_1 > r_1'$ and $h \in \mathrm{b}\mathcal{S}$ such that

$$B(h, r_1) \subset B(f, R) \setminus A\,. \tag{2}$$

We shall show that there exists a function $g \in \mathrm{b}\mathcal{S}$ such that

$$B\Big(g, \frac{r_1}{4}\Big) \subset B(h, r_1) \setminus \mathcal{L}\,. \tag{3}$$

Let $x_0 \in (0,1)$ be a continuity point of $h$ (see Theorem 6) and let $\delta \in \big(0, \frac{1}{2}\min(x_0, 1-x_0)\big)$ be an arbitrary real number such that

$$h\big([x_0-\delta,\, x_0+\delta]\big) \subset \Big(h(x_0) - \frac{r_1}{4},\, h(x_0) + \frac{r_1}{4}\Big)\,. \tag{4}$$

We shall consider a Cantor-like set $\mathbf{C} \subset [x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}]$ such that $x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2} \in \mathbf{C}$ and $\lambda(\mathbf{C}) > 0$. Then there exists a non-measurable set $\mathbf{C_1} \subset \mathbf{C}$ ([5]) such that $x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2} \in \mathbf{C} \setminus \mathbf{C}_1$. Let $\{(a_n, b_n)\}_{n\in\mathbb{N}}$ be a sequence of all components of the complement of $\mathbf{C}$.

Let us fix $n \in \mathbb{N}$ and let $g_n\colon (a_n, b_n) \to \mathbb{R}$ be a continuous function such that

$$g_n\big((a_n, x)\big) = \Big[-\frac{r_1}{4}, \frac{r_1}{4}\Big] = g_n\big((x, b_n)\big)\,, \quad \text{for an arbitrary } x \in (a_n, b_n)\,. \tag{5}$$

Now, we shall define the function $g\colon [0,1] \to \mathbb{R}$ in the following way:

$$g(x) = \begin{cases} h(x) & \text{if } \in [0,1] \setminus (x_0 - \delta,\, x_0 + \delta)\,, \\ h(x_0 - \delta) & \text{if } x \in \big[x_0 - \delta,\, x_0 - \frac{3}{4}\delta\big]\,, \\ l_1(x) & \text{if } x \in \big[x_0 - \frac{3}{4}\delta,\, x_0 - \frac{\delta}{2}\big]\,, \\ \frac{r_1}{4} + h(x_0) & \text{if } x \in \mathbf{C} \setminus \mathbf{C_1}\,, \\ -\frac{r_1}{4} + h(x_0) & \text{if } x \in \mathbf{C_1}\,, \\ g_n(x) + h(x_0) & \text{if } x \in (a_n, b_n) \text{ for } n = 1, 2, \dots\,, \\ l_2(x) & \text{if } x \in \big[x_0 + \frac{\delta}{2},\, x_0 + \frac{3}{4}\delta\big]\,, \\ h(x_0 + \delta) & \text{if } x \in \big[x_0 + \frac{3}{4}\delta,\, x_0 + \delta\big]\,, \end{cases}$$

where $l_1$ is a linear function such that $l_1\big(x_0 - \frac{3}{4}\delta\big) = h(x_0 - \delta)$ and $l_1\big(x_0 - \frac{\delta}{2}\big) = \frac{r_1}{4} + h(x_0)$; $l_2$ is a linear function such that $l_2\big(x_0 + \frac{\delta}{2}\big) = \frac{r_1}{4} + h(x_0)$ and $l_2\big(x_0 + \frac{3}{4}\delta\big) = h(x_0 + \delta)$.

We will show that

$$g \in \mathrm{b}\mathcal{S}\,. \tag{6}$$

Obviously, $g$ is a bounded function. Now we will prove that

$$g\Big|\big[x_0 - \tfrac{\delta}{2},\, x_0 + \tfrac{\delta}{2}\big] \text{ is the strong Świątkowski function}\,. \tag{7}$$

For this purpose we shall use the "classical" definition of the strong Świątkowski function (in the global sense) ([3]).

Let $x, y \in \big[x_0 - \frac{\delta}{2},\, x_0 + \frac{\delta}{2}\big]$ be real numbers such that $x < y$ and $g(x) \neq g(y)$. Let $\alpha \in \big(g(x), g(y)\big)$. Assume, for instance, that $g(x) < \alpha < g(y)$. We shall show that

$$\text{there exists } z \in (x, y) \cap \mathcal{C}(g) \text{ such that } g(z) = \alpha\,. \tag{8}$$

Then we shall consider the following cases:

1) Let $x, y \in (a_n, b_n)$ where $(a_n, b_n)$ is a component of the complement of the Cantor set. Since $g|(a_n, b_n)$ is a continuous function for each $n \in \mathbb{N}$, there exists $z \in (x, y) \cap \mathcal{C}(g)$ such that $g(z) = \alpha$, which proves the condition (8).

2) Let $x \in \mathbf{C}_1$, $y \in \mathbf{C} \setminus \mathbf{C}_1$ (or $x \in \mathbf{C} \setminus \mathbf{C}_1$, $y \in \mathbf{C}_1$). Since the complement of the Cantor set is dense, $(x, y) \setminus \mathbf{C} \neq \emptyset$. Thus there exists $z_0 \in (x, y) \setminus \mathbf{C}$. So,

$z_0 \in (a_{n_0}, b_{n_0})$ for some $n_0 \in \mathbb{N}$. Then $(a_{n_0}, b_{n_0}) \subset (x,y)$ (because $x, y \in \mathbf{C}$). From the fact that $g\big|(a_{n_0}, b_{n_0})$ is a continuous function, $g\big((a_{n_0}, b_{n_0})\big) = \big[h(x_0) - \frac{r_1}{4}, h(x_0) + \frac{r_1}{4}\big]$ and $\alpha \in \big(g(x), g(y)\big) = \big(h(x_0) - \frac{r_1}{4}, h(x_0) + \frac{r_1}{4}\big)$ there exists $z \in (a_{n_0}, b_{n_0}) \subset (x,y) \cap \mathcal{C}(g)$ such that $g(z) = \alpha$, finishing the proof of (8) in this case.

3) Let $x \in \mathbf{C}_1$, $y \in (a_n, b_n)$ for some $n \in \mathbb{N}$ (if $x \in \mathbf{C} \setminus \mathbf{C}_1$, $y \in (a_n, b_n)$ or $x \in (a_n, b_n)$, $y \in \mathbf{C}_1$ or $x \in (a_n, b_n)$, $y \in \mathbf{C} \setminus \mathbf{C}_1$ the proof is analogous). Then $(a_n, y) \subset (x, y)$. From the definition of the function $g$, we have

$$g\big((a_n, y)\big) = \Big[h(x_0) - \frac{r_1}{4},\, h(x_0) + \frac{r_1}{4}\Big].$$

Moreover, $\alpha \in \big(g(x), g(y)\big) \subset \big(-\frac{r_1}{4} + h(x_0), \frac{r_1}{4} + h(x_0)\big) \subset \big[h(x_0) - \frac{r_1}{4}, h(x_0) + \frac{r_1}{4}\big]$ and $g|(a_n, y)$ is a continuous function. Thus there exists $z \in (a_n, y) \subset (x, y) \cap \mathcal{C}(g)$ such that $g(z) = \alpha$. This completes the proof of (8). So the proof of (7) is finished.

Moreover, it is easy to show that $x_0 - \delta$, $x_0 - \frac{\delta}{2}$, $x_0 + \frac{\delta}{2}$, $x_0 + \delta$ do not cut the function $g$.

Thus from Lemma 8, (7) and the fact that $g\big|\big[x_0 - \delta,\, x_0 - \frac{\delta}{2}\big]$, $g\big|\big[x_0 + \frac{\delta}{2},\, x_0 + \delta\big]$, $g|[0, x_0 - \delta] = h|[0, x_0 - \delta]$, $g|[x_0 + \delta, 1] = h|[x_0 + \delta, 1]$ are compatible strong Świątkowski functions we infer that $g \in \mathrm{b}\mathcal{S}$. So, the proof of (6) is complete.

Moreover, according to (4) we have:

$$\rho(g, h) \le \frac{r_1}{2}.$$

Now, let us notice that

$$B\Big(g, \frac{r_1}{4}\Big) \subset B(h, r_1). \tag{9}$$

We will show that

$$B\Big(g, \frac{r_1}{4}\Big) \cap \mathcal{L} = \emptyset. \tag{10}$$

Let $\psi \in B(g, \frac{r_1}{4})$. Then

$$\psi^{-1}\big((-\infty, h(x_0))\big) \cap \mathbf{C} = \mathbf{C_1}.$$

Indeed, if $x \in \mathbf{C_1}$, $\psi(x) < g(x) + \frac{r_1}{4} = -\frac{r_1}{4} + h(x_0) + \frac{r_1}{4} = h(x_0)$ and $x \in \psi^{-1}\big((-\infty, h(x_0))\big) \cap \mathbf{C}$.

On the other hand, let $x \in \psi^{-1}\big((-\infty, h(x_0))\big) \cap \mathbf{C}$. Thus $\psi(x) < h(x_0)$ and $x \in \mathbf{C}$. Since $g(x) - \frac{r_1}{4} < \psi(x)$, then $g(x) - \frac{r_1}{4} < h(x_0)$, $g(x) < h(x_0) + \frac{r_1}{4}$ and $x \in \mathbf{C}$. Hence, by the definition of the function $g$, we have $x \in \mathbf{C_1}$.

If $\psi^{-1}\big((-\infty, h(x_0))\big)$ is a measurable set, then obviously $\psi^{-1}\big((-\infty, h(x_0))\big) \cap \mathbf{C} = \mathbf{C_1}$ is a measurable set, too. Therefore $\psi^{-1}\big((-\infty, h(x_0))\big)$ is a non-measurable set. Thus we have proved that $\psi \notin \mathcal{L}$. This finishes the proof of (10).

From (9) and (10) it follows (3). From (3) and (2) we infer that

$$B\Big(g, \frac{r_1}{4}\Big) \subset B(f, R) \setminus (A \cup \mathcal{L}).$$

Hence

$$\gamma(f, R, A \cup \mathcal{L}) = \sup\big\{r > 0 \colon \exists_{\xi \in \mathrm{b}\mathcal{S}}\, B(\xi, r) \subset B(f, R) \setminus (A \cup \mathcal{L})\big\} \geq \frac{r_1}{4}.$$

So, it is easy to see that the set $A \cup \mathcal{L}$ is porous at $f$. The proof of the theorem is complete. □

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