Tatra Mt. Math. Publ. **23** (2001), 1–15

TATRA MOUNTAINS
Mathematical Publications

ON SOME PROPERTIES OF PERIODIC SEQUENCES IN ANATOL VIERU'S MODAL THEORY

MORENO ANDREATTA — DAN T. VUZA

ABSTRACT. Algebraic methods have been currently applied to music in the second half of the twentieth-century (see [M. Andreatta: *Group-theoretical Methods applied to Music*, unpublished dissertation, 1997], [M. Chemilier: *Structure et Méthode algébraiques en informatique musicale.* Thèse de doctorat, L. I. T. P., Institut Blaise Pascal, 1990] and [G. Mazzola et al.: *The Topos of Music—Geometric Logic of Concepts, Theory and Performance*] for main references). By starting from Anatol Vieru's compositional technique based on finite difference calculus on periodic modal sequences, as it has been introduced in his book [*Cartea modurilor, 1* (*Le livre des modes, 1*). Ed. Muzicala, Bucarest, 1980. Revised ed. *The book of modes*, 1993], the present essay tries to generalize some properties by means of abstract group theory. Two main classes of periodic sequences are considered: reducible and reproducible sequences, replacing respectively Vieru's modal and irreducible sequences. It turns out that any periodic sequence can be decomposed in a unique way into a reducible and a reproducible component.

1. Reducible and reproducible sequences

For any sequence f defined on $\mathbb{Z}$ taking values in a finite abelian group G we define the translated sequence Tf and the sequence of differences Df by

$$Tf(x) = f(x+1)\,, \quad Df(x) = f(x+1) - f(x)\,.$$

The relationship between the translated sequence and the sequence of differences is expressed by the following equation:

$$D = T - 1\,.$$

2000 Mathematics Subject Classification: 20C11, 39A10, 39A70.
Keywords: reducible, reproducible, rarefied and repeating sequences, decomposition.

The content of the following paper has been presented by one of the authors at I. R. C. A. M. (Institut de Recherche et de Coordination Acoustique/Musique) during the special session on algebraic formalization of musical structures organized by the Music Representations Team within the context of the Séminaires Entretemps *Musique, Mathématiques et Philosophie* (under the direction of G. Assayag, F. Nicolas and G. Mazzola). We want to express our thanks to Charlotte and Florent for helping us in making up this paper.

Definition 1. The sequence f is called *m-periodic* if $f(x+m)=f(x)$ for any $x\in\mathbb{Z}$.

Affirming that f is m-periodic, it is equivalent to the relation $T^m f = f$. If f is m-periodic, then Tf and Df are also m-periodic.

Definition 2. The sequence f is called *reducible* if an integer $k\geq 0$ does exist such that $D^k f = 0$.

The sequence f is called *reproducible* if an integer $k\geq 0$ does exist such that $D^k f = f$.

By $\mathrm{Red}(G)$ and $\mathrm{Rep}(G)$ we will designate respectively the set of reducible and reproducible sequences taking values in G (also called G-valued sequences).

Example 1. Example of a reducible sequence. In Anatol Vieru's book [10] a $\mathbb{Z}_{12}$-valued sequence of period 72 is considered. It can be represented by means of 6 lines of 12 elements

$$\begin{array}{cccccccccccc}
(4 & 1 & 0 & 8 & 8 & 7 & 0 & 6 & 8 & 1 & 4 & 0\\
8 & 11 & 4 & 6 & 0 & 5 & 4 & 4 & 0 & 11 & 8 & 10\\
0 & 9 & 8 & 4 & 4 & 3 & 8 & 2 & 4 & 9 & 0 & 8\\
4 & 7 & 0 & 2 & 8 & 1 & 0 & 0 & 8 & 7 & 4 & 6\\
8 & 5 & 4 & 0 & 0 & 11 & 4 & 10 & 0 & 5 & 8 & 4\\
0 & 3 & 8 & 10 & 4 & 9 & 8 & 8 & 4 & 3 & 0 & 2).
\end{array}$$

Any column is a periodic sequence obtained by adding modulo 12 a constant value to a basis element (i.e., the top of the column). By putting the constant value as an index for the basis element, one has the following compact expression

$$(4_4\ 1_{10}\ 0_4\ 8_{10}\ 8_4\ 7_{10}\ 0_4\ 6_{10}\ 8_4\ 1_{10}\ 4_4\ 0_{10}),$$

$$D^1 = (2_6\ 9_6\ 11_6\ 8_6\ 0_6\ 11_6\ 5_6\ 6_6\ 2_6\ 5_6\ 3_6\ 8_6),$$

$$D^2 = (0\ 7\ 2\ 9\ 4\ 11\ 6\ 1\ 8\ 3\ 10\ 5),$$

$$D^3 = (7),$$

$$D^4 = 0.$$

Example 2. Example of a reproducible sequence

$$f = (8\ 11\ 0\ 1\ 4\ 0),$$

$$D^1 = (3\ 1\ 1\ 3\ 8\ 8),$$

$$D^2 = (10\ 0\ 2\ 5\ 0\ 7),$$

$$D^3 = (2\ 2\ 3\ 7\ 7\ 3),$$

$$D^4 = f.$$

Theorem 1. *Let d_i, $0 \leq i \leq N$, be some integers such that at least one of them is relatively prime to the number of elements of G. Then, an integer m does exist such that any sequence verifying*

$$\sum_{i=0}^{N} d_i T^i f = 0$$

is m-periodic.

Corollary 2. *Reducibles and reproducibles sequences are periodic.*

Theorem 3. *All periodic sequence can be decomposed in a unique way*

$$f = f_{\text{red}} + f_{\text{rep}}\,, \qquad f_{\text{red}} \in \text{Red(G)}\,, \;\; f_{\text{rep}} \in \text{Rep(G)}\,.$$

Proof. Let f be m-periodic. Being the collection of sequences m-periodic finite, two integers $k, l \geq 1$ do exist such that $D^k f = D^{k+l} f$. By induction on r one has $D^k f = D^{k+rl} f$. In the same way it can be shown that two integers $r, s \geq 1$ may exist such that $D^{rl} f = D^{(r+s)l} f$. We put

$$f_{\text{red}} = f - D^{rl} f, \qquad f_{\text{rep}} = D^{rl} f\,.$$

It follows that $D^k(f - D^{rl} f) = 0$, $D^{sl} D^{rl} f = D^{rl} f$, which means that f_{red} and f_{rep} give the needed decomposition.

The unicity comes from the relation $\text{Red(G)} \cap \text{Rep(G)} = \{0\}$. □

2. Decomposition of $\mathbb{Z}_n$ into p-groups

There is a one-to-one relation between the subgroups of $\mathbb{Z}_n$ and the family of integers d that divide n by $1 \leq d \leq n$, i.e., for any such d we may take the unique subgroup of $\mathbb{Z}_n$ with d elements. The latter can be characterized as the set of $z \in \mathbb{Z}_n$ such that $dz = 0$ or, equivalently, as the set $\frac{n}{d}\mathbb{Z}_n$ of elements having the form $\frac{n}{d} z$ where z belongs to $\mathbb{Z}_n$.

Definition 3. The abelian group G is a *direct sum of a family of subgroups* $G_1, \ldots, G_m$ of G if any $x \in G$ may be decomposed in a unique way into a sum $x_1 + \cdots + x_m$ with $x_i \in G_i$ for $1 \leq i \leq m$.

We will put $G = \bigoplus_{i=1}^{m} G_i$.

Definition 4. Let p be a prime number. A finite abelian group is called *p-group* if its cardinality is a power of p.

THEOREM 4. *Any group $\mathbb{Z}_n$ is a direct sum of its p-maximals subgroups.*

If $n = \prod_{i=1}^{m} p^{k_i}$ is the decomposition of n into prime factors, the decomposition of $\mathbb{Z}_n$ in maximals p-subgroups can be written as follows

$$\mathbb{Z}_n = \bigoplus_{i=1}^{m} G_{p^{k_i}},$$

where $G_{p^{k_i}}$ is the subgroup of $\mathbb{Z}_n$ with p^{k_i} elements. The decomposition of any $z \in \mathbb{Z}_n$ defines the elements $\pi_i(z) \in G_{p^{k_i}}$ in a unique way such that $z = \sum_{i=1}^{m} \pi_i(z)$. The arrows $\pi_i \colon \mathbb{Z}_n \to G_{p^{k_i}}$ are group morphisms.

The p_i-component $\pi_i(z)$ of z is the unique element $y \in \mathbb{Z}_n$ satisfying the relations $p^{k_i} y = \frac{n}{p^{k_i}}(z-y) = 0$ in $\mathbb{Z}_n$. Let q_i be an integer verifying

$$q_i \frac{n}{p^{k_i}} = 1 \bmod p^{k_i}\,.$$

It follows that $p^{k_i} \frac{n}{p^{k_i}} q_i z = \frac{n}{p^{k_i}}(z - \frac{n}{p^{k_i}} q_i z) = 0$ in $\mathbb{Z}_n$, which means that

$$\pi_i(z) = \frac{n}{p^{k_i}} q_i z\,.$$

EXAMPLE 3. Subgroups of $\mathbb{Z}_{12}$

G_1	$\{0\}$,	
G_2	$\{0,\ 6\}$,	
G_3	$\{0,\ 4,\ 8\}$,	maximal 3-group,
G_4	$\{0,\ 3,\ 6,\ 9\}$,	maximal 2-group,
G_6	$\{0,\ 2,\ 4,\ 6,\ 8,\ 10\}$,	
G_{12}	$\{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11\}$.	

EXAMPLE 4. Decomposition of $\mathbb{Z}_{12}$ into p-groups

$$\mathbb{Z}_{12} = G_3 \bigoplus G_4\,.$$

	0	3	6	9
0	0	3	6	9
4	4	7	10	1
8	8	11	2	5

$$q_3 = 1\,, \quad 1 \cdot \frac{12}{3} = 1 \bmod 3\,,$$
$$q_4 = -1\,, \quad -1 \cdot \frac{12}{4} = 1 \bmod 4\,,$$
$$\pi_3(x) = \frac{12}{3}\, q_3 x = 4x\,, \quad \pi_4(x) = \frac{12}{4}\, q_4 x = -3x\,.$$

$$5 = 8 + 9 = 4 \cdot 5 - 3 \cdot 5 = 4 \cdot 2 - 3 \cdot 1\,,$$
$$7 = 4 + 3 = 4 \cdot 7 - 3 \cdot 7 = 4 \cdot 1 - 3 \cdot 3\,,$$
$$11 = 8 + 3 = 4 \cdot 11 - 3 \cdot 11 = 4 \cdot 2 - 3 \cdot 3\,.$$

Remark 1. Many theoretical works ([2], [8] and [10]) have already shown the musical interest of the decomposition of the cyclic group $\mathbb{Z}_{12}$ into a direct sum of its maximal p-groups from a theoretical, analytical and compositional point of view.

Theorem 5. *Let $\varphi_i \colon \mathbb{Z}_n \to \mathbb{Z}_{p^{k_i}}$ be the canonic map. The map $\varphi \colon \mathbb{Z}_n \to \prod_{i=1}^{m} \mathbb{Z}_{p^{k_i}}$ defined by*

$$\varphi(z) = \big(\varphi_1(z), \ldots, \varphi_m(z)\big)$$

is a ring homomorphism with inverse given by

$$(z_1, \ldots, z_m) \mapsto \sum_{i=1}^{m} \frac{n}{p^{k_i}}\, q_i z_i\,.$$

For any $\mathbb{Z}_n$-valued sequence f we put $f_i(x) = \pi_i\big(f(x)\big)$. We will say that $f = \sum_i f_i$ is the decomposition of f corresponding to the decomposition of $\mathbb{Z}_n$ in p-groups.

3. Characterization of reducible sequences

Proposition 6. *Let $f = \sum_j f_j$ be the decomposition of f corresponding to the decomposition of $\mathbb{Z}_n$ in p-groups. Then $f \in \mathrm{Red}(\mathbb{Z}_n)$ if and only if $f_j \in \mathrm{Red}(\mathbb{Z}_n)$ for any j.*

Theorem 7. *Let f be a $\mathbb{Z}_{p^k}$-valued sequence. Then f is reducible if and only if it is p^m-periodic for a given $m \geq 0$.*

P r o o f. By induction on k. For $k = 1$ the result comes from the following lemma. ☐

Lemma 8. *For any $\mathbb{Z}_p$-valued sequence f, one has*

$$(T-1)^{p^m} f = \left(T^{p^m} - 1\right) f\,.$$

P r o o f. By induction on m one is conducted to the case $m = 1$.

$$(T-1)^p f = \sum_{i=0}^{p} (-1)^{p-i} C_p^i T^i f\,.$$

Since p divides any C_p^i by $1 \leq i \leq p-1$ and since $pf = 0$ it follows that the sum of the right member is reduced to $T^p f - f$.

Let now the theorem be true for any $k_1 < k$. We designate f_1 the sequence defined by $f_1(x) = pf(x)$. It is clear that f_1 takes values in the subgroup $G_{p^{k-1}}$ with p^{k-1} elements of $\mathbb{Z}_{p^k}$, isomorphic to $\mathbb{Z}_{p^{k-1}}$.

Let f be p^m-periodic. Then f_1 is also p^m-periodic, i.e., reducible (induction hypothesis). Therefore, there exists $l \geq 1$ such that $D^l f_1 = 0$. By definition of $f_{1,}$ it means that $p(D^l f)(x) = 0$ for any x, i.e., the sequence $D^l f$ takes values into the subgroup G_p with p elements of $\mathbb{Z}_{p^k}$, isomorphic to $\mathbb{Z}_p$. Since $D^l f$ is p^m-periodic, the induction hypothesis leads to the existence of $l_1 \geq 1$ such that $D^{l_1} D^l f = 0$, i.e., $D^{l_1 + l} f = 0$ and f is reducible.

Let f be reducible. Then $f_1 \in \mathrm{Red}(G_{p^{k-1}})$ by induction hypothesis it follows that there exists $m_1 \geq 0$ such that f_1 is p^{m_1}-periodic. This means that $f(x) - f(x + p^{m_1}) \in G_p$ for any x. We define the sequence f_2 by $f_2(x) = f(x) - f(x + p^{m_1})$. Since $f_2 \in \mathrm{Red}(G_p)$, the induction hypothesis guaranties the existence of $m_2 \geq 0$ such that f_2 is p^{m_2}-periodic. In short, $(1 - T^{p_{m_2}})(1 - T^{p_{m_1}})f = 0$. To conclude we just need the following. □

Lemma 9. *Let f be a $\mathbb{Z}_n$-valued sequence such that $\left(1 - T^k\right)\left(1 - T^l\right) f = 0$. Then f is kln-periodic.*

P r o o f. From $\left(1 - T^l\right) f = T^k \left(1 - T^l\right) f$ one deduces that $\left(1 - T^l\right) f = T^{ki}\left(1 - T^l\right) f$ for any $i \geq 1$, and in particular that $\left(1 - T^{kl}\right)\left(1 - T^l\right) f = 0$. By the same argument, $\left(1 - T^{kl}\right) f = T^{li}\left(1 - T^{kl}\right) f$ for any $i \geq 1$. This means that

$$\left(1 - T^{kln}\right) f = \left(1 - \left(T^{kl}\right)^n\right) f = \left(\sum_{i=0}^{n-1} T^{kli}\right)\left(1 - T^{kl}\right) f = n\left(1 - T^{kl}\right) f = 0\,.$$

□

Corollary 10. $\mathrm{Red}(\mathbb{Z}_n)$ *is a ring.*

Corollary 11. *Let $f \in \mathrm{Red}(\mathbb{Z}_n)$ be such that $f(x)$ is invertible in $\mathbb{Z}_n$ for all x. Then $f^{-1} \in \mathrm{Red}(\mathbb{Z}_n)$.*

Corollary 12. *Let k, l be two integers and let g be defined by $g(x) = f(kx + l)$. If $f \in \mathrm{Red}(\mathbb{Z}_n)$ then $g \in \mathrm{Red}(\mathbb{Z}_n)$.*

4. Characterization of reproducible sequences

Proposition 13. *Let $f = \sum_j f_j$ be the decomposition of f corresponding to the decomposition of $\mathbb{Z}_n$ in p-groups. Then $f \in \mathrm{Rep}(\mathbb{Z}_n)$ if and only if $f_j \in \mathrm{Rep}(\mathbb{Z}_n)$ for any j.*

Definition 5. For every m-periodic sequence f and for every integer d dividing m, we define the *d-periodised* obtained from f as the sequence $\sum\limits_{i=0}^{m/d-1} T^{id} f$.

The equation

$$(1 - T^d) \sum_{i=0}^{m/d-1} (T^d)^i f = (1 - (T^d)^{m/d}) f = (1 - T^m) f = 0$$

shows that the d-periodised sequence is indeed a d-periodic sequence.

Theorem 14. *Let f be a $\mathbb{Z}_{p^k}$-valued m-periodic sequence. Then f is reproducible if and only if the p^r-periodised of f is zero, where p^r is the highest power of p that divides m.*

P r o o f. If f is reproducible, then its p^r-periodised is also reproducible by construction, but also reducible according to Theorem 7, because it is p^r-periodic. Then the p^r-periodised is zero, because $\mathrm{Rep} \cap \mathrm{Red} = \{0\}$.

Conversely, by supposing that the p^r-periodised is zero, we may write the decomposition

$$f = f_{\mathrm{red}} + f_{\mathrm{rep}}, \qquad f_{\mathrm{red}} \in \mathrm{Red}, \; f_{\mathrm{rep}} \in \mathrm{Rep}.$$

According as we have shown, the p^r-periodised of f_{rep} is zero. Then the previous equation shows that the p^r-periodised of f_{red} must be zero too. Due to Theorem 7, f_{red} must be p^s-periodised for a given integer $s \geq 0$. Because of the fact that the m-periodised of f implies that of f_{red}, the period of the latter divides m and p^s, which shows that f_{red} is in fact p^r-periodic. As a consequence its p^r-periodised is equal to $\frac{m}{p^r} f_{\mathrm{red}}$. Since $\frac{m}{p^r} \wedge p^k = 1$, the multiplication by $\frac{m}{p^r}$ is an automorphism of $\mathbb{Z}_{p^k}$. From $\frac{m}{p^r} f_{\mathrm{red}} = 0$ one may conclude that $f_{\mathrm{red}} = 0$.

□

Corollary 15. *Let f be a $\mathbb{Z}_{p^k}$-valued m-periodic sequence and let p^r be the highest power of p dividing m. Then f is reproducible if and only if the relations*

$$\sum_{i=0}^{m/p^r-1} f(p^r i + x) = 0$$

hold for all x such that $0 \leq x < p^r$.

5. Calculation of reducible and reproducible components of a periodic sequence

Proposition 16. *Let $f = \sum_j f_j$ be the decomposition of the periodic sequence f corresponding to the decomposition of $\mathbb{Z}_n$ in p-groups and let $f_j = f_{j,\mathrm{red}} + f_{j,\mathrm{rep}}$ be the decomposition of f_j in a sum of a reducible and a reproducible sequence. Then the decomposition of f*

$$f = f_{\mathrm{red}} + f_{\mathrm{rep}}\,, \qquad f_{\mathrm{red}} \in \mathrm{Red}\,,\ f_{\mathrm{rep}} \in \mathrm{Rep}\,,$$

is given by

$$f_{\mathrm{red}} = \sum_j f_{j,\mathrm{red}}\,,\ f_{\mathrm{rep}} = \sum_j f_{j,\mathrm{rep}}\,.$$

Theorem 17. *Let f be a $\mathbb{Z}_{p^k}$-valued m-periodic sequence and let f_{per} be the p^r-periodised of f, where p^r is the highest power of p dividing m. Let $\left(\frac{m}{p^r}\right)^{-1}$ be the inverse of $\frac{m}{p^r}$ $\bmod p^k$. Then the decomposition of f*

$$f = f_{\mathrm{red}} + f_{\mathrm{rep}}\,, \qquad f_{\mathrm{red}} \in \mathrm{Red}\,,\ f_{\mathrm{rep}} \in \mathrm{Rep}\,,$$

is given by

$$f_{\mathrm{red}} = \left(\frac{m}{p^r}\right)^{-1} f_{\mathrm{per}}\,, \qquad f_{\mathrm{rep}} = f - \left(\frac{m}{p^r}\right)^{-1} f_{\mathrm{per}}\,.$$

Proof. Since f_{red} is p^r-periodic by construction, it is reducible according to the Theorem 7. The p^r-periodised of f_{rep} is zero:

$$\sum_{i=0}^{m/p^r-1} T^{ip^r} f_{\mathrm{rep}} = \sum_{i=0}^{m/p^r-1} T^{ip^r} f - \left(\frac{m}{p^r}\right)\left(\frac{m}{p^r}\right)^{-1} f_{\mathrm{per}} = f_{\mathrm{per}} - f_{\mathrm{per}} = 0\,.$$

Due to the Theorem 14, f_{rep} is reproducible. □

Corollary 18. *Let m and n be two integers such that $m \wedge n = 1$ and let f be a $\mathbb{Z}_n$-valued m-periodic sequence. f is reproducible if and only if $\sum\limits_{i=0}^{m-1} f(i) = 0$.*

6. Decomposition algorithm

1. Write the decomposition of n

$$n = \prod_{j=1}^{N} p_j^{k_j}$$

into prime factors.

2. Find the integers q_j such that
$$q_j \frac{n}{p_j^{k_j}} = 1 \mod p_j^{k_j}.$$
3. For all j build the sequences $f_{j,\mathrm{red}}$ and $f_{j,rep}$ as follows.
4. Set $f_j(x) = \varphi_j\big(f(x)\big)$, where $\varphi_j : \mathbb{Z}_n \to \mathbb{Z}_{p_j^{k_j}}$ is the canonic map.
5. Let $p_j^{r_j}$ be the highest power of p_j dividing the period m of f. Determine the inverse $\left(\frac{m}{p_j^{r_j}}\right)^{-1}$ of $\frac{m}{p_j^{r_j}}$ mod $p_j^{k_j}$.
6. Write down the m elements of the period of f_j as a table with $\frac{m}{p_j^{r_j}}$ lines and $p_j^{r_j}$ columns (if $r_j = 0$, the table will have an unique column). Add a line given by the elements of any column and by multiplying by $\left(\frac{m}{p_j^{r_j}}\right)^{-1}$ module $p_j^{k_j}$.
7. Let $f_{j,\mathrm{red}}$ be the $p_j^{r_j}$-periodic sequence defined by the line built at the previous step and build $f_{j,\mathrm{rep}} = f_j - f_{j,\mathrm{red}}$.
8. By setting
$$f_{\mathrm{red}} = \sum_{i=1}^{N} q_j \frac{n}{p_j^{k_j}} f_{j,\mathrm{red}}, \qquad f_{\mathrm{rep}} = \sum_{i=1}^{N} q_j \frac{n}{p_j^{k_j}} f_{j,\mathrm{rep}}.$$
we have the decomposition of any periodic sequence in a reducible and reproducible component.

Example 5. Decomposition in reducible and reproducible components. Let f be the following $\mathbb{Z}_{12}$-valued sequence.
$$f = (0\ 0\ 7\ 7\ 4\ 4\ 3\ 3\ 4\ 4\ 7\ 7).$$
The sequences corresponding to the decomposition of $\mathbb{Z}_{12} = G_3 \bigoplus G_4$ are respectively
$$f_1 = (0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 1\ 1)$$
$$f_2 = (0\ 0\ 3\ 3\ 0\ 0\ 3\ 3\ 0\ 0\ 3\ 3)$$

	0	0	1
	1	1	1
	0	0	1
	1	1	1
	2	2	1
$\div 4$ mod 3	2	2	1
$\times 4$ mod 12	8	8	4

	0	0	3	3
	0	0	3	3
	0	0	3	3
	0	0	1	1
$\div\, 3 \bmod 4$	0	0	3	3
$\times -3 \bmod 12$	0	0	3	3

	8	8	4	8	8	4	8	8	4	8	8	4
	0	0	3	3	0	0	3	3	0	0	3	3
f_{red}	8	8	7	11	8	4	11	11	4	8	11	7
f_{rep}	4	4	0	8	8	0	4	4	0	8	8	0

7. The sets of values of reducible and reproducible sequences

THEOREM 19. *Let $\mathbb{Z}_n = \bigoplus_j G_j$ be the decomposition of $\mathbb{Z}_n$ in p-groups. A subset M of $\mathbb{Z}_n$ is the set of values of a reducible sequence if and only if two subsets $M_j \subset G_j$ do exist such that $M = \sum_j M_j$.*

This theorem is a direct consequence of the the following proposition.

PROPOSITION 20. *Let $\mathbb{Z}_n = \bigoplus_j G_j$ be the decomposition of $\mathbb{Z}_n$ in p-groups and let $f = \sum_j f_j$ be the decomposition of the reducible sequence f corresponding to the decomposition of $\mathbb{Z}_n$. Then $f(\mathbb{Z}) = \sum_j f_j(\mathbb{Z})$.*

P r o o f. Clearly $f(\mathbb{Z}) \subset \sum_j f_j(\mathbb{Z})$. On the other hand, let $y_j = f_j(x_j) \in f_j(\mathbb{Z})$ and, for all integer p_j dividing n, let p_j^r be the highest power of p_j dividing the period m of f. According to the Theorem 7, f_j is p_j^r-periodic. A well known result of algebra gives the existence of $x \in \mathbb{Z}$ verifying $x = x_j \bmod p_j^r$ for all j. It follows that $f_j(x) = f_j(x_j)$ for all j, i.e.,

$$\sum_j y_j = \sum_j f_j(x_j) = \sum_j f_j(x) = f(x) \in f(\mathbb{Z}).$$

□

THEOREM 21. *For any subset $M \subset \mathbb{Z}_n$ it does exist an element $u \in \mathbb{Z}_n$ and a reproducible sequence f such that $f(\mathbb{Z}) = u + M$.*

P r o o f. It does exist an integer m such that $m \wedge n = 1$ and the m-periodic sequence g such that $g(\mathbb{Z}) = M$. Since $m \wedge n = 1$, the inverse m^{-1} of m mod n does exist. We set $u = -m^{-1} \sum_{i=0}^{m-1} g(i)$ and $f(x) = u + g(x)$. It follows that $\sum_{i=0}^{m-1} f(i) = 0$, which gives the reproducibility of f due to the Corollary 18.

□

Remark 2. Generally, it does not exist a reproducible sequence f such that $f(\mathbb{Z}) = M$. For example, take $M = \{3, 9\} \subset \{0, 3, 6, 9\} \subset \mathbb{Z}_{12}$. By supposing that a reproducible f may exist such that $f(\mathbb{Z}) = M$, let $m = 2^r d$ be the period of f, where d is odd. By Theorem 14 one has $\sum_{i=0}^{d-1} f(2^r i) = 0$. Since $f(2^r i) \in \{3, 9\}$, it follows that some integers k_1, k_2 must exist such that $3k_1 + 9k_2 = 0 \mod 12$, $k_1 + k_2 = d$. This means that

$$\begin{aligned} 3k_1 + 9(d - k_1) &= 0 \mod 12\,, \\ 6k_1 + 9d &= 0 \mod 12\,, \\ 2 \cdot 9d = -2 \cdot 6k_1 &= 0 \mod 12\,, \\ 3d &= 0 \mod 2\,. \end{aligned}$$

But because of the oddness of d one would conclude that $3 = 0 \mod 2$, which is absurd.

Example 6. Modal classes of sets of values of $\mathbb{Z}_{12}$-valued reducible sequences

$$\begin{array}{llllll}
\{0\} & + & \{0\} & = & \{0\}\,, \\
\{0\} & + & \{6,6\} & = & \{6,6\}\,, \\
\{0\} & + & \{9,3\} & = & \{9,3\}\,, \\
\{0\} & + & \{6,3,3\} & = & \{6,3,3\}\,, \\
\{0\} & + & \{3,3,3,3\} & = & \{3,3,3,3\}\,, \\
\{8,4\} & + & \{0\} & = & \{8,4\}\,, \\
\{8,4\} & + & \{6,6\} & = & \{4,2,4,2\}\,, \\
\{8,4\} & + & \{9,3\} & = & \{5,3,1,3\}\,, \\
\{8,4\} & + & \{6,3,3\} & = & \{1,2,1,3,2,3\}\,, \\
\{8,4\} & + & \{3,3,3,3\} & = & \{1,2,1,2,1,2,1,2\}\,, \\
\{4,4,4\} & + & \{0\} & = & \{4,4,4\}\,, \\
\{4,4,4\} & + & \{6,6\} & = & \{2,2,2,2,2,2\}\,, \\
\{4,4,4\} & + & \{9,3\} & = & \{3,1,3,1,3,1,3,1\}\,, \\
\{4,4,4\} & + & \{6,3,3\} & = & \{1,1,2,1,1,2,1,1,2\}\,, \\
\{4,4,4\} & + & \{3,3,3,3\} & = & \{1,1,1,1,1,1,1,1,1,1,1,1\}\,.
\end{array}$$

8. The reproducibility of rarefied and repeating sequences

Let f be a periodic $\mathbb{Z}_n$-valued sequence and let d be a positive integer.

Definition 6. The *rarefied sequence* f_{rar} obtained from the sequence f is given by the following relation:

$$f_{\mathrm{rar}}(x) = \begin{cases} f\left(\frac{x}{d}\right) & \text{if } x = 0 \mod d\,, \\ 0 & \text{if } x \neq 0 \mod d\,. \end{cases}$$

The integer d is called the *factor of insertion.*

Definition 7. The *repeating sequence* obtained from f is the sequence f_{rpt} defined by

$$f_{\mathrm{rpt}}(x) = f\left(\left[\frac{x}{d}\right]\right)$$

where $[r]$ means the highest integer $\leq r$. The integer d is called the *factor of repetition.*

We will write $f_{\mathrm{rar},d}$ and $f_{\mathrm{rpt},d}$ when we want to make evident the presence of the factor d. It follows that

$$\begin{aligned} f_{\mathrm{rar},d_1 d_2} &= (f_{\mathrm{rar},d_1})_{\mathrm{rar},d_2}\,, \\ f_{\mathrm{rpt},d_1 d_2} &= (f_{\mathrm{rpt},d_1})_{\mathrm{rpt},d_2}\,. \end{aligned} \tag{1}$$

Problem. Find the relationship between the reproducibility of f, f_{rar} and f_{rpt}.

Lemma 22.

$$f_{\mathrm{rpt}} = \sum_{i=0}^{d-1} T^i f_{\mathrm{rar}}\,.$$

P r o o f. This result is a trivial consequence of the definition of rarefied and repeating sequences. □

Definition 8. A sequence f is called d-reproducible if $(T^d - 1)^m f = 0$ for an integer $m \geq 1$.

Lemma 23. *If f is a d-reproducible sequence then f is reproducible.*

P r o o f. Let f be d-reproducible. We write down the decomposition

$$f = f_{\mathrm{red}} + f_{\mathrm{rep}}\,, \qquad f_{\mathrm{red}} \in \mathrm{Red}\,,\ f_{\mathrm{rep}} \in \mathrm{Rep}\ .$$

It follows that $(T^d - 1)^m f = f$ and $(T-1)^m f_{\mathrm{red}} = D^m f_{\mathrm{red}} = 0$. According to the identity

$$T^d - 1 = P(T)(T-1)\,, \qquad P(T) = \sum_{i=0}^{d-1} T^i, \tag{2}$$

one has that

$$\begin{aligned} f &= (T^d - 1)^m f = P(T)^m (T-1)^m f \\ &= P(T)^m (T-1)^m f_{\mathrm{red}} + P(T)^m (T-1)^m f_{\mathrm{rep}} \\ &= P(T)^m (T-1)^m f_{\mathrm{rep}} \in \mathrm{Rep}\ . \end{aligned}$$

□

THEOREM 24. *The reproducibility of f is equivalent to that of f_{rar}.*

Proof. If f is reproducible, the relation

$$(T^d - 1)f_{\mathrm{rar}} = \big((T-1)f\big)_{\mathrm{rar}}$$

shows that f_{rar} is d-reproducible, i.e., reproducible by the Lemma 23.

Now let suppose that f_{rar} is reproducible. Then $f = \sum_j f_j$ is the decomposition of f corresponding to the decomposition of $\mathbb{Z}_n$ in p-groups. Since the decomposition of f_{rar} is $\sum_j (f_j)_{\mathrm{rar}}$, it is enough to consider the case where $n = p^k$, p prime. According to (1), we do not loose generality by taking d which is prime. By writing down the decomposition

$$f = f_{\mathrm{red}} + f_{\mathrm{rep}}\,, \qquad f_{\mathrm{red}} \in \mathrm{Red}\,,\ f_{\mathrm{rep}} \in \mathrm{Rep}\,,$$
$$f_{\mathrm{rar}} = (f_{\mathrm{red}})_{\mathrm{rar}} + (f_{\mathrm{rep}})_{\mathrm{rar}}\,.$$

one has that $f_{\mathrm{rar}} \in \mathrm{Rep}$ and $(f_{\mathrm{rep}})_{\mathrm{rar}} \in \mathrm{Rep}$ because of i), i.e., $(f_{\mathrm{red}})_{\mathrm{rar}} \in \mathrm{Rep}$ and one is led to show that if $f \in \mathrm{Red}(\mathbb{Z}_{p^k})$ and $f_{\mathrm{rar}} \in \mathrm{Rep}$, then $f = 0$. Now, since f is reducible, it has to be p^m-periodic. Therefore f_{rar} is dp^m-periodic. If $d = p$ then f_{rar} is reducible too, i.e., $f_{\mathrm{rar}} \in \mathrm{Rep} \cap \mathrm{Red} = \{0\}$ which means that $f = 0$. If $d \neq p$, the reproducibility of f_{rar} is expressed by the relations

$$\sum_{i=0}^{d-1} f_{\mathrm{rar}}(ip^m + j) = 0\,, \qquad 0 \le j < p^m. \tag{3}$$

The only terms different from zero in the left side of (3) are those for which

$$ip^m + j = 0 \mod d\,. \tag{4}$$

Since $d \wedge p = 1$, the equation (4) in the variable i has an unique solution $i(j)$ for all j. It follows that in the left part of (3), just one term is zero, i.e.,

$$f_{\mathrm{rar}}(i(j)p^m + j) = f\left(\frac{i(j)p^m + j}{d}\right).$$

This means that

$$f\left(\frac{i(j)p^m + j}{d}\right) = 0\,, \qquad 0 \le j < p^m. \tag{5}$$

The numbers $\big(i(j)p^m + j\big)/d$, $0 \le j < p^m$, form a fundamental system of classes mod p^m. In fact, if

$$\frac{i(j_1)p^m + j_1}{d} = \frac{i(j_2)p^m + j_2}{d} \mod p^m,$$

then

$$i(j_1)p^m + j_1 = i(j_2)p^m + j_2 \mod p^m,$$

i.e., $j_1 = j_2 \mod p^m$ which means that $j_1 = j_2$. According to (5) and to the p^m-periodicity of f, one obtains $f = 0$. $\square$

Lemma 25. *If $d \wedge n = 1$ and if $\sum_{i=0}^{d-1} T^i f$ is reproducible then f is reproducible.*

P r o o f . Let's write down the decomposition

$$f = f_{\mathrm{red}} + f_{\mathrm{rep}}\,, \qquad f_{\mathrm{red}} \in \mathrm{Red}\,,\ f_{\mathrm{rep}} \in \mathrm{Rep}\,,$$

$$P(T)f = P(T)f_{\mathrm{red}} + P(T)f_{\mathrm{rep}}\,,$$

where $P(T)$ is given by (2). Since $P(T)f \in \mathrm{Rep}$ (by hypothesis) and also $P(T)f_{\mathrm{rep}} \in \mathrm{Rep}$, one has $P(T)f_{\mathrm{red}} \in \mathrm{Red} \cap \mathrm{Rep} = \{0\}$. This leads to the relation

$$(T^d - 1)f_{\mathrm{red}} = (T-1)P(T)f_{\mathrm{red}} = 0\,,$$

which means that f_{red} is d-periodic. Since f_{red} is reducible, it is also m-periodic where m is an integer that divides n. This means that f is $d \wedge m$-periodic, which gives the 1-periodicity, i.e., f is a constant. In conclusion

$$f = z + f_{\mathrm{rep}}\,, \qquad z \in \mathbb{Z}_n\,,\ f_{\mathrm{rep}} \in \mathrm{Rep}\,,$$

$$P(T)f = dz + P(T)f_{\mathrm{rep}}\,.$$

By using the same argument $dz \in \mathrm{Red} \cap \mathrm{Rep} = \{0\}$, i.e., $z = 0$ since the multiplication by d is an automorphism of $\mathbb{Z}_n$. □

Theorem 26. *The reproducibility of f is equivalent to that of f_{rpt}.*

P r o o f . If f is reproducible, then (Theorem 24) f_{rar} is reproducible, which means that (Lemma 22) f_{rpt} is reproducible, too.

Let's now suppose that f_{rpt} is reproducible. As in the proof of Theorem 24 one may just consider the case where f takes its values in a p- group and d is prime. If $d = p$ one uses the same argument as in the proof of Theorem 24. If $d \neq p$, then (Lemmas 22 and 25) f_{rar} is reproducible, which leads to the conclusion that (Theorem 24) f is reproducible too. □

REFERENCES

[1] ANDREATTA, M.: *Group-theoretical Methods applied to Music*, unpublished Dissertation, University of Sussex, 1997.

[2] BALZANO, G.: *The group-theoretic description of 12-fold and microtonal pitch systems*, Computer Music J. **4** (1980), 66–84.

[3] BUDDEN, F. J.: *The Fascination of Groups*, Cambridge University Press, 1972.

[4] CHEMILLIER, M.: *Structure et Méthode algébriques en informatique musicale*, (Thèse de doctorat), L. I. T. P., Institut Blaise Pascal (1990).

[5] HALSEY, D.—HEWITT, E.: *Eine gruppentheoretische Methode in der Musik-theorie*, Jahresber. Deutsch. Math.-Verein. **80** (1978), 151–207.

[6] LEWIN, D.: *Generalized Musical Intervals and Transformations*, Yale University Press, 1987.

[7] MAZZOLA, G. et. al.: *The Topos of Music—Geometric Logic of Concepts, Theory and Performance*, Birkäuser (to appear).

[8] MAZZOLA, G.: *A Symmetry-Oriented Mathematical Model of Classical Counterpoint and Related Neurophysiological Investigations by Depth-EEG*, Dans Symmetry II, CAMWA, Pergamon Press, 1989.

[9] STEIN, S. K.—SZABÓ, S.: *Algebra and Tiling*, The Carus Mathematical Monographs, Vol. 25, AMS, Providence, 1994.

[10] VIERU, A.: *Cartea modurilor, 1* (*Le livre des modes, 1*) (Muzicala, Bucarest, 1980, ed.), Revised ed. *The book of modes*, 1993.

[11] VUZA, D. T.: *Some Mathematical Aspects of David Lewin's Book Generalized Musical Intervals and Transformations*, Perspectives of New Music **26** (1988), 258–287.

[12] VUZA, D. T.: *Supplementary Sets — Theory and Algorithms*, Muzica **1** (1995), 75–99.

[13] VUZA, D. T.: *Aspects mathématiques dans la théorie modale d' Anatol Vieru, Parts 1–4*, Revue Roumaine de Mathématiques Pures et Appliquées **27**, (1982) n. 2 et 10, (1983) n. 7 et 8.

Received February 1, 2001

Moreno Andreatta
Equipe Représentations Musicales
Ircam/CNRS — Centre G. Pompidou
Paris
FRANCE
E-mail: Moreno.Andreatta@ircam.fr

Dan T. Vuza
Department of Mathematics
University of Bucharest
Bucharest
ROMANIA
E-mail: Dan.Vuza@imar.ro